STOCHASTIC RESONANCES OF QUANTUM QUARTIC OSCILLATOR

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Abstract. Quantum oscillator with spatial quartic potential and parametric complement has been investigated in context of the Schrödinger-Langevin-Kostin equation. The properties and stochastic resonances are numerically studied by means of the time realizations for mean coordinate and its Fourier spectra. Analogous effects take place in an empty well.

Keywords: friction, quantum quartic oscillator, stochastic resonance

1. Introduction

Stochastic resonances in distinct fields such as astrophysics, radio-electronics, informatics and biology have been studied extensively more than three decades. First, the term “stochastic resonance” was introduced in scientific literature in Ref. [1] where modelling of the Earth climate was performed. Then analogous phenomena were investigated for an electron stream in the dynamical system with irregular distribution of positive ion density [2]. The review of papers related to the stochastic resonances in macroscopic dynamical systems is given in Ref. [3]. This paper is concerned with the quantum microscopic dynamical system which is exposed to a cubic classical force, friction and Langevin white noise. The numerical investigation has been carried out in the context of the Schrödinger-Langevin-Kostin (SchLK) formalism.

2. Basic equations and designations

The equation of quantum motion in a non-dimensional form is formulated as

$$i \frac{\partial \psi}{\partial \tau} = \left( -\frac{1}{2} \frac{\partial^2}{\partial \zeta^2} + U_L + U_{\text{diss}} + U_{\text{rand}} \right) \psi,$$

being invariant relative to a choice of measurement units. Equation (1) was introduced in Ref. [4, 5] and named as the Schrödinger-Langevin-Kostin equation (SchLK). The equation defines the time evolution of the wave function $\psi(\zeta, \tau)$, where $\zeta, \tau$ are position and time, respectively, that satisfies to the normalization condition

$$\int_{-\zeta_L}^{\zeta_L} \psi^* \psi d\zeta = 1.$$  \hspace{1cm} (2)

The quantum system is confined by impenetrable walls in the points $\pm \zeta_L$, where $\zeta_L$ is a half-width of the system. The boundary conditions on the walls and the initial conditions are specified as:

$$\psi(\pm \zeta_L, \tau) = 0, \psi(\zeta, \tau = 0) = \psi_0(\zeta).$$  \hspace{1cm} (3)
Here $\psi_0(\zeta)$ is the solution of stationary Schrödinger equation for a ground state. The oscillations are excited by the single short-time pulse potential

$$U_0(\zeta, \tau) = -\mathcal{F}_0(\zeta), \quad \tau \in (0, \Delta \tau_0),$$

where $\mathcal{F}_0$ is the force acting for a time $\Delta \tau_0$. The quantity $\Delta \tau_0$ is much less than the characteristic oscillation period.

In Eq. (1) the quantity $U_\Sigma$ defines the potential consisting of

$$U_\Sigma = U_\nu + U_{\text{ext}},$$

where $U_\nu = \zeta^4$, $U_{\text{ext}} = U_0 \sin(\Omega_{\text{ext}} \tau) \zeta^4$. $U_{\text{ext}}$ is the external (or parametric) portion; $U_0, \Omega_{\text{ext}}$ are amplitude and frequency.

For the dissipation potential $U_{\text{diss}}$ we have

$$U_{\text{diss}} = -\frac{ik}{2} \left( \ln \frac{\psi}{\psi^*} - \left\langle \ln \frac{\psi}{\psi^*} \right\rangle \right), \quad \left\langle \ln \frac{\psi}{\psi^*} \right\rangle = \int \psi^* \ln \frac{\psi}{\psi^*} \psi d\zeta.$$  

Here $k$ is a friction coefficient, $\psi^*$ is the complex conjugate wave function.

The Langevin stochastic potential $U_{\text{rand}}$ is considered as

$$U_{\text{rand}} = -\mathcal{F}_{\text{rand}}(\tau) \zeta^4,$$

where $\mathcal{F}_{\text{rand}} = D \sqrt{2/\Delta \tau} \text{erf}^{-1}(2r-1)$ is the random force corresponding to a white noise; $D$ is a dispersion and $r$ is the random number from 0 to 1; $\Delta \tau$ is a numerical time step. The method of the random Langevin force $\mathcal{F}_{\text{rand}}$ in form (7), worked out in Ref. [6], was applied in Ref. [7]. The quantum wave-packet dynamics was investigated using the distribution of probability density $N = \psi^* \psi$, the mean values of position $\langle \zeta \rangle$ and velocity $\langle V \rangle$, and the standard deviations $\sigma_\zeta = \sqrt{\langle (\Delta \zeta)^2 \rangle}$, $\sigma_V = \sqrt{\langle (\Delta V)^2 \rangle}$. The product $\sigma_\zeta \sigma_V$ defines the uncertainty relation, it being analyzed.

The frequency spectra for the time evolution of $\langle \zeta \rangle$ were studied by the fast Fourier transform method, the value of modulus square $F_{\langle \zeta \rangle}(\Omega) = |\Phi_{\langle \zeta \rangle}(\Omega)|^2$ as a function of frequency $\Omega$ being investigated.

### 3. Stochastic resonances

For interpreting the dynamical behavior, it is necessary to know the energy spectra of the stationary Schrödinger equation for spatially confined quantum system without friction. The calculations of energy spectra and transition frequencies between neighboring levels were performed for the well of half width equal to $\zeta_L = 3$ (Table 1).

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_n$</td>
<td>0.668</td>
<td>2.394</td>
<td>4.697</td>
<td>7.336</td>
<td>10.24</td>
<td>13.38</td>
<td>16.71</td>
<td>20.22</td>
</tr>
<tr>
<td>$\Delta \varepsilon_n$</td>
<td>1.726</td>
<td>2.303</td>
<td>2.639</td>
<td>2.907</td>
<td>3.135</td>
<td>3.333</td>
<td>3.509</td>
<td>3.669</td>
</tr>
</tbody>
</table>
As a first step, we investigated the free oscillations generated by the action of a single pulse on the ground state without an external harmonic force and friction. The single pulse duration was equal to 0.5 and the pulse amplitude \( F_0 \) was varied. In Fig. 1 we see the simple picture of oscillations.

![Map of probability density](image1)

(a) Map of probability density \( N \)

![Time realization of \( \zeta \)](image2)

(b) Time realization of \( \zeta \)

![Fourier transform \( F_{\zeta}(\Omega) \)](image3)

(c) Fourier transform \( F_{\zeta}(\Omega) \)

![Uncertainty product \( \sigma_\zeta \sigma_\Omega \)](image4)

(d) Uncertainty product \( \sigma_\zeta \sigma_\Omega \)

**Fig. 1.** Free oscillations of a quartic oscillator at \( F_0 = -1 \)

The most simple behavior and oscillatory patterns are realized at small \( F_0 \). In this case the map of probability density \( N \) on the plane \( (\zeta, \tau) \) takes form like a fir (Fig. 1a), a dark color corresponds to larger values of \( N \). The time realization of \( \zeta \) and its Fourier transform \( F_{\zeta}(\Omega) \) are presented in Figs. 1b, c. Here two spectral components at \( \Omega_1 = 1.7257 \) and \( \Omega_2 = 2.3032 \) are excited; the most extensive spectral component is realized at \( \Omega_1 \) with \( F_{\zeta}(\Omega_1) \approx 0.5 \), the spectral component at \( \Omega_2 \) is small and equal to \( F_{\zeta}(\Omega_2) \approx 0.06 \). The difference \( \Omega_2 - \Omega_1 \) determines a shallow amplitude modulation which is well seen in time realization of \( \zeta \). The oscillatory patterns become more complicated with increasing \( |F_0| \) due to the generation of new Fourier components.

Below we discuss the problem of stochastic resonances under a white noise. To eliminate the spectral component on a frequency of \( \Omega_2 \), it is necessary to decrease \( |F_0| \) to 0.1. Thus the only one spectral component \( F_{\zeta}(\Omega_1) \) on a frequency of \( \Omega_1 = 1.7257 \) is generated. It is equal to the frequency of transition from the ground state into the first excited
one. Other spectral components are not generated. Turning on $\mathcal{F}_{\text{rand}}$ with small value of dispersion $D$ modifies the dynamical evolution, the spectral component $F_{\langle \zeta ^2 \rangle}(\Omega)$ is amplified that is caused by resonance. In addition, the spectral component on a frequency of $\Omega = 6.65$ arises too; it corresponds to one of a number of transition frequencies for an empty well. Hence, at small values of $\mathcal{F}_{\text{rand}}$ two spectral components prevail, the remaining components are weak and represent a broad-band spectrum. The dynamical properties change with increasing $D$, as it is shown in Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Stochastic resonances in a quantum quartic oscillator}
\end{figure}

The discrete components $F_{\langle \zeta ^2 \rangle}(\Omega)$ and the broad-band spectrum are enhanced in proportion to $\mathcal{F}_{\text{rand}}$ (about ten times). In addition, the spectral components corresponding to the higher frequencies of transition (between first and second excited states and so on) such as $\Omega = 2.3032, \Omega = 2.639, \ldots$ are also generated.

For the case of the empty quantum well with $U_{\text{st}} = U_{\text{ext}} = 0, k = 0$ and $\mathcal{F}_{\text{rand}} \neq 0$, the quantum wave evolution is represented in Fig. 3. Figures 3b, d demonstrate a plethora of separated spectral peaks correspondent to the transition frequencies of empty well. The amplitudes of peaks and broad-band components are decreased with increasing a frequency.
In the regime, the number of separate spectral peaks was small; the shutdown of $U_{st}$ brings into existence the multitude of separated peaks caused by stochastic resonances.

The friction is destructive to resonant spectral peaks for above-enumerated parameters. Here the calculations were performed for some friction coefficients, when the resonant spectral components disappear on the background of broad-band noise.

In Figure 4 the dynamical regimes are presented for parameters: $D = 31.6, k = 1$. It should be noted that the $F_{\zeta}(\Omega)$ for quantum quartic oscillator is characterized by a visible maximum in the vicinity of $\Omega$. In second case with the quantum empty well, we have the monotonous attenuation of broad-band noise.

4. Quantum parametric quartic oscillator with a noise

Introducing an parametric term into the expression for potential $U_\Sigma$ opens new dynamical properties, the simplest consists in generation of combination frequencies. For the quartic oscillator with the parametric frequency $\Omega_\text{ext} = \Omega_1/4$ at the varied noise level from $D = 3.2$ to 31.6 and $U_0 = 0.1$, the resonance takes place. The spectral components $F_{\zeta}(\Omega_1)$, $F_{\zeta}(\Omega_1 \pm \Omega_\text{ext})$ and spectral component $F_{\zeta}(\Omega_\alpha)$ on a frequency of $\Omega_\alpha = 6.65$ of the empty well.
well are amplified (Table 2).

The spectral components do not change while \( 0 < D < 0.32 \), after the threshold value \( D = 0.32 \) the rise of separate components occurs (Table 2). Increasing the components \( F_{\zeta}(\Omega_1) \), \( F_{\zeta}(\Omega_1 \pm \Omega_{\text{ext}}) \) takes place in the region \( 0.32 < D < 25.5 \); at \( D > 25.5 \) it is confined, but the component \( F_{\zeta}(\Omega_a) \) grows in \((0.32, 31.6)\). The components of other combined frequencies have been absorbed.

If the friction is introduced into the system, the dynamical situation changes cardinally. At \( k = 0.1 \) the spectral component at the frequency of transition from the ground state into the first exciting one is attenuated almost ten-fold, the other components being absorbed in a broadband noise.

![Graphs showing the absorption of separate resonant components due to friction](image)

Fig. 4. Absorption of separate resonant components due to friction

Table 2. Spectral components of a quantum quartic oscillator

<table>
<thead>
<tr>
<th>( D )</th>
<th>( F_{\zeta}(\Omega_1) )</th>
<th>( F_{\zeta}(\Omega_1 - \Omega_{\text{ext}}) )</th>
<th>( F_{\zeta}(\Omega_a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.32</td>
<td>( 2 \times 10^{-2} )</td>
<td>( 2 \times 10^{-3} )</td>
<td>( 3 \times 10^{-5} )</td>
</tr>
<tr>
<td>3.2</td>
<td>( 9 \times 10^{-2} )</td>
<td>( 4 \times 10^{-3} )</td>
<td>( 1.5 \times 10^{-4} )</td>
</tr>
<tr>
<td>31.6</td>
<td>( 1.5 \times 10^{-1} )</td>
<td>( 10^{-2} )</td>
<td>( 10^{-3} )</td>
</tr>
</tbody>
</table>
5. Summary
A quantum anharmonic oscillator with quartic potential, ohmic friction and white noise has been discussed as a generalization of the classical analogue. Using the Schrödinger-Langevin-Kostin equation, the temporal realizations of the mean dynamical variables and their frequency spectra were investigated numerically. As the illustrative pattern, for the isolated quantum quartic oscillator (without friction and noise) under a weak short pulse excitement; a single spectral component is generated, corresponding to a frequency of transition from ground state into the first excited one. The other spectral components are absent or very small.

For the open quantum system including the oscillator and a white noise, the dynamical behavior is radically changed. Already at small noise dispersion (weak Langevin force), the spectral component on a frequency of transition from the ground state to the first excited one is essentially augmented due to resonance. In addition, the spectral component on a frequency of empty well transition (without quartic potential) is generated. At large noise dispersion (strong Langevin force), the new spectral components of higher order frequency transitions are generated. The spectral component on the frequency of transition from the first excited state into the second one becomes comparable with the basic spectral component.

The quantum quartic oscillator is sensitive to friction. The numerical results are given for the friction coefficient less than the transition frequency. In this regime, the broad-band maximum takes place at the basic frequency. The analogous dynamical phenomena occur with quantum wave packet in the empty potential well (without quartic potential). The white noise initiates also the resonant growth of the single discrete spectral components. Due to the friction, these spectral components are absorbed and disappear from the region of the frequency spectrum.

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