Fatigue fracture of thin rectangular plates with central crack under uniaxial high-cyclic symmetrical loading

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Abstract

There are various criteria for determining crack extension conditions which play an important role in the study of fatigue fracture. The paper presents an approach based on the phenomenological viewpoint which uses continuum mechanics concepts. The criterion of fatigue crack growth as due to the microdamage accumulation in the vicinity of a crack tip is considered. The approach has been proposed in [1] to the construction of fatigue fracture models for infinite plates with cracks. A special emphasis has been made on high-cycle loading which do not produce any significant macroplastic strains, while the stage of the stable growth of a crack may occupy a dominant proportion of the fatigue lifetime.

In this paper the approach is used to construct the generalized two-stage fatigue crack growth model for thin finite plates under high-cyclic uniaxial symmetrical loading. The model makes it possible to take into account the incubation stage as well as propagation stage of cracks growth and includes the function that estimates the effect of finiteness of the plate dimensions and cracks length. One of advantages of the model is the fact that the coefficients of the model are material constants to be determined from two base experiments of plain cylindrical specimens.

Within the framework of the developed models some problems of fatigue cracks growth in thin isotropic finite plates for two different materials have been solved. The results calculated on the basis of developed two-stage model agree well with those obtained by experiment.

Keywords: finite plate, fatigue crack, damage, cyclic plastic zone.

1 Formulation of the problem and background

Consider a normal tensile crack with initial half-length l_0 in a thin finite isotropic plate with width W and height h in plane stress state (Fig. 1). The plate is subjected to uniform uniaxial completely reversed loading $\tilde{\sigma}$ applied to the ends of the plate and can be represented in the form

$$\tilde{\sigma} = \sigma_a \cdot g\left(n\right), \tag{1}$$

where σ_a is amplitude stress, $g(\cdot)$ is a known alternating function of a number of cycles n of a change in stresses ($n = ft$), t is the physical time, and f is the loading frequency. It is also assumed in (1) that, the applied stress amplitude σ_a is not time dependent (stationary regime), varies quite often ($f > 10\,\text{Hz}$), and the maximum stress in a cycle does not exceed the yield stress σ_Y of a material (high cycle fatigue). In this case the main body of a plate is deformed linear-elastically and fatigue fracture is quasibrittle.

The crack surfaces are taken as entirely free from applied force and contact interaction of crack faces is not taken into account.

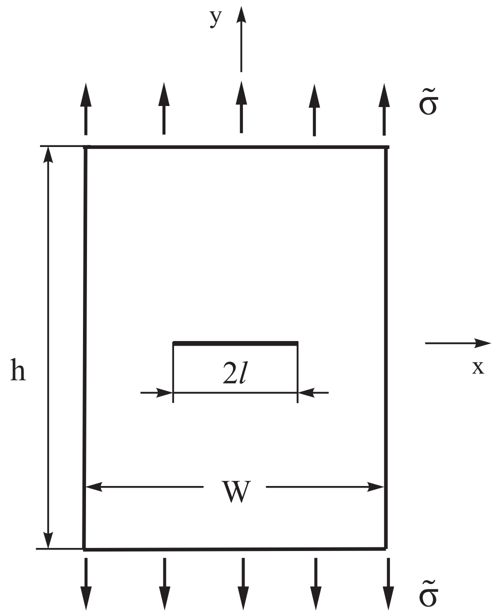

Figure 1: Plate geometry and load scheme

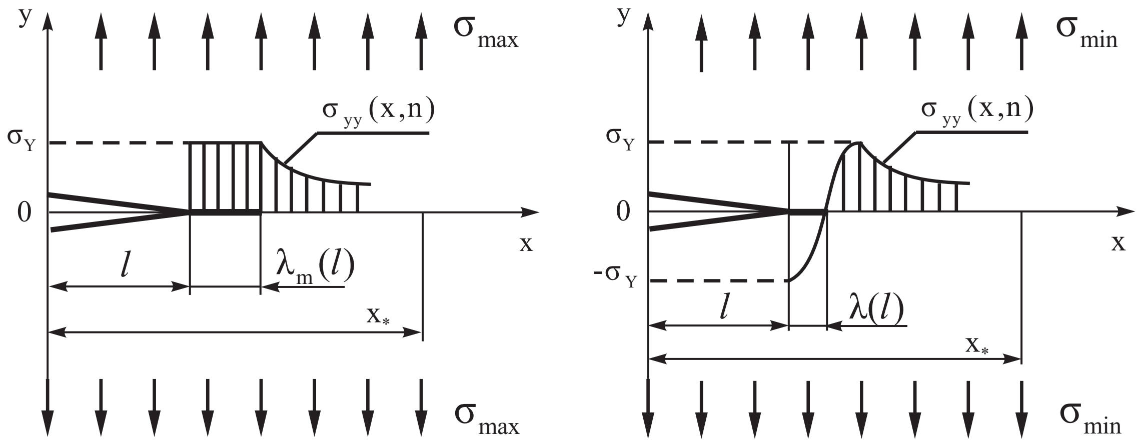

Figure 2: Crack-closure model in cyclic loading.

The fatigue crack is considered as a narrow, sharp-tipped slot. The modified Dugdale model [2, 3] is used to describe two different plastic zones formed at the crack tip (Fig. 2). One of which $\lambda_m(l)$ is a monotonic plastic zone which occurs in the half-cycle of tension and stress in the zone is limited by the yield stress σ_Y . Another one $\lambda(l)$ is cyclic plastic zone formed in the half-cycle of compression and stress in the zone reaches $-\sigma_Y$. Outside of the plastic zone the material of a plate is deformed linear-elastically. It is supposed also that the crack is growing along the x - axis and the center of the crack is at $x = 0$.

The problem is to derive a two-stage model of the fatigue crack growth in a thin finite plate which gives the moment of the crack movement starting of length l_0 and determines crack length l and crack length change rate l versus the number of cycles of loading n .

In fact the problem can be reduce to specify the relation

$$\frac{dl}{dn} = F(\sigma_a,\ l,\ n,\ h,\ W,\ C_i), \tag{2}$$

or

$$l = F_1 \left(\sigma_a,\ n,\ h,\ W,\ C_i\right), \tag{3}$$

where C_i are material constants $\left(i = \overline{1,k}\right)$.

Fatigue crack growth model is based on joint solving of the problem of determination of the local stress field at the moving crack tip and the problem of formulation of criterion of fatigue crack propagation using the concepts of damage mechanics. The set of corresponding simultaneous equations includes equilibrium equations

$$\begin{cases} \dfrac{\partial \tilde{\sigma}_{xx}}{\partial x} + \dfrac{\partial \tilde{\tau}_{yx}}{\partial y} = 0 \\ \dfrac{\partial \tilde{\tau}_{xy}}{\partial x} + \dfrac{\partial \tilde{\sigma}_{yy}}{\partial y} = 0 \end{cases}, \tag{4}$$

compatibility equations in the terms of stress

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\tilde{\sigma}_{xx} + \tilde{\sigma}_{yy}\right) = 0, \tag{5}$$

evolution equation of the accumulation of fatigue damage with initial and boundary values

$$\begin{cases} \dfrac{\partial \omega\left(x,n\right)}{\partial n} = D\left(\dfrac{\Delta\sigma\left(x,n\right)}{1-\omega\left(x,n\right)}\right)^q \\ \omega\left(x,n\right) = \begin{cases} 0 & at\ n = 0 \\ 1 & at\ n = n_R \end{cases} \end{cases}, \tag{6}$$

and the condition of local failure at the crack front

$$\begin{cases} \max\left\{\omega\left[l_0 + \lambda\left(l_0\right), n\right]\right\} = 1 & 0 \le n \le n_* \\ \max\left\{\omega\left[l\left(n\right) + \lambda\left(l\left(n\right)\right), n\right]\right\} = 1 & n \ge n_* \end{cases}, \tag{7}$$

where $\tilde{\sigma}_{xx},\ \tilde{\sigma}_{yy},\ \tilde{\tau}_{yx}$ are cyclic stresses tensor components, ω is scalar damage parameter, n_R is number of cycles to failure, $\Delta\sigma$ is stress range at crack front, n_* is certain critical number of loading cycles, D and q are material constants to be determined experimentally, $l\left(n\right)$ is the current length of crack at any time.

Governing equations are supplemented by boundary conditions

$$\begin{gathered} \tilde{\sigma}_{yy} = \begin{cases} \pm\sigma_a & y = \pm\dfrac{h}{2} & -\dfrac{W}{2} \le x \le \dfrac{W}{2} \\ 0 & y = 0 & \begin{cases} -l_0 \le x \le l_0 & 0 \le n \le n_* \\ l\left(n\right) \le x \le l\left(n\right) & n > n_* \end{cases} \end{cases} \\ \tilde{\sigma}_{xx} = \quad 0 \qquad x = \pm\frac{W}{2} \qquad -\frac{h}{2} \le y \le \frac{h}{2}, \end{gathered} \tag{8}$$

which correspond with a given geometry of the plate with a crack.

2 Stress distribution at the crack front

The improved method of collocation [4] is used for the stress analysis of finite plate with central crack. The numerical method is combined on the complex variable method of [5] and modified boundary-collocation method. Using of the approach provides obtaining of exact solution in the interior of the plate and approximate satisfaction of boundary conditions.

The method is based on application biharmonic Airy stress function $\Phi(x, y)$ which can be expressed as

$$\Phi(x, y) = Re\left[\int_S \overline{\Psi(z)} d\bar{z} + \int_S \Omega(z)\, dz + (\bar{z} - z)\,\Omega(z)\right], \tag{9}$$

where $\Psi'(z) = \phi(z)$, $\Omega'(z) = \chi(z)$, $\phi(z)$ and $\chi(z)$ are complex potentials, $z = x + iy$ and $\bar{z} = x - iy$ are complex variables, S is contour of integration.

Then at any time stresses tensor components can be represented in the form

$$\sigma_{xx} = \frac{\partial^2 \Phi(x, y)}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi(x, y)}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi(x, y)}{\partial x \partial y}. \tag{10}$$

In this case the equilibrium and compatibility equation are combined to equation

$$16 \frac{\partial^4 \Phi(z, \bar{z})}{\partial z^2 \partial \bar{z}^2} = 0. \tag{11}$$

The boundary conditions are transformed as

$$2 \frac{\partial \Phi(z, \bar{z})}{\partial \bar{z}} = \int \overline{\phi(z)} d\bar{z} + \int \chi(z)\, dz + (z - \bar{z})\, \overline{\chi(z)} = f_1 + i f_2, \tag{12}$$

where f_1 and f_2 are forces acting on the boundary.

Complex potentials $\phi(z)$ and $\chi(z)$ are specified to provide stress free conditions on the crack surfaces [4]

$$\begin{cases} \phi(z) = \dfrac{z}{\sqrt{z^2 - l^2}} \displaystyle\sum_{n=0}^{N} A_n z^{2n} + \sum_{n=0}^{N} B_n z^{2n} \\ \chi(z) = \dfrac{z}{\sqrt{z^2 - l^2}} \displaystyle\sum_{n=0}^{N} A_n z^{2n} - \sum_{n=0}^{N} B_n z^{2n} \end{cases}, \tag{13}$$

where A_n and B_n - series of coefficients determined by satisfying boundary conditions at the collocation points, N - is the number of coefficients in the series which stress function is resolved by.

Stress components are expressed in terms of complex potentials $\phi(z)$ and $\chi(z)$ by equations

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 2\left[\chi(z) + \overline{\chi(z)}\right], \\ \sigma_{yy} - \sigma_{xx} - 2i\tau_{xy} &= 2\left[(z - \bar{z})\, \overline{\chi'(z)} - \overline{\chi(z)} + \overline{\phi(z)}\right]. \end{aligned} \tag{14}$$

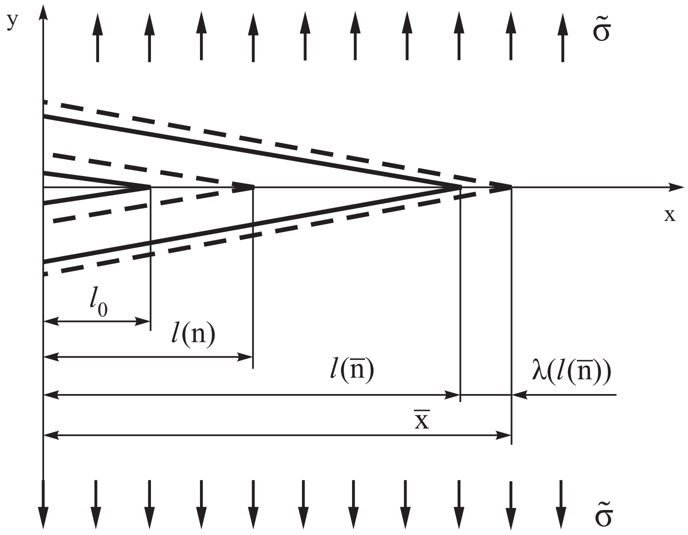

Figure 3: Fatigue crack growth.

From (14) with taking into account (13) we obtain the expression for stress range along Ox -axis ($y = 0,\ z = x$) at the cyclic loading

$$\Delta\sigma_{yy} = \frac{\sigma_a}{\sqrt{2}}\sqrt{\frac{l}{x-l}} \cdot \left(1 + \sum_{n=1}^{N} 2A_n x^{2n}\right), \tag{15}$$

where we neglect the compression portion of the loading cycle, because it does not effect significantly the crack growth [6].

Using polynomial approximation of numerical solution (15) stress distribution at the crack tip along failure front takes the following form

$$\Delta\sigma_{yy}\left(x,n\right) = \frac{\sigma_a}{\sqrt{2}}\sqrt{\frac{l\left(n\right)}{x-l\left(n\right)}} \cdot f\left(\frac{h}{W}, \frac{l\left(n\right)}{W}\right), \tag{16}$$

where $f\left(\frac{h}{W}, \frac{l(n)}{W}\right)$ is correcting function that estimates the effect of finiteness of the plate dimensions and initial crack length.

3 Model of fatigue crack growth

The fatigue fracture is considered as process of the fatigue damage accumulation. The damage accumulation mechanism is complicated, therefore it is often described with the help of continual approach [7].

The damage level is specified by a scalar damage function $\omega(x,n)$, which accumulation kinetics is given by the differential equation (6). Assume $\omega\left(x,0\right) = 0$ at initial time $n = 0$, the movement of starting and propagation of fatigue crack occurs when the fatigue damage on fracture front at point $\bar{x}$ (Fig. 3) in the time n reaches the value of unity.

The fatigue fracture of a cracked plate is considered further as a two-stage process which consists of the incubation stage and crack propagation stage. By the incubation stage we will mean a certain period of loading of n_* time while the repeated opening and closing of crack surfaces occur without an increase in its initial length. The crack propagation stage is related to the new surfaces formation and to the dependence of the crack length l on the number of cycles of loading n .

Integrated (6) with taking into account the two-stage character of the fatigue fracture process, initial and fracture conditions we obtain integral equation for fatigue crack front movement

$$\int_0^1 \left[1-\omega\left(x_*,n\right)\right]^q d\omega = D\left[\int_0^{n_*}\left[\sigma_{yy}\left(\bar{x},n\right)\right]^q dn + \int_{n_*}^{n}\left[\sigma_{yy}\left(\bar{x},n\right)\right]^q dn\right], \tag{17}$$

where n_* is the duration of the incubation stage.

Suggested that crack length l at fracture moment increases on length of cyclic plastic zone $\lambda\left(l\right)$, let'swrite

$$\bar{x} = \left[\begin{array}{ll} l_0+\lambda(l_0), & 0\le n\le n_* \\ l(\bar{n})+\lambda(l(\bar{n})), & n_*< n<\bar{n} \end{array}\right., \tag{18}$$

Taking into account (16), equation (17) is transformed to non-linear integral equation Volterra of the first kind

$$\begin{gathered} 1-(1+q)D\left(\frac{\sigma_a}{\sqrt{2}}\right)^q\int_0^{n_*}\left[\frac{l_0}{l(\bar{n})+\lambda(l(\bar{n}))-l_0}\right]^{\frac{q}{2}}\cdot f\left(\frac{h}{W},\frac{l_0}{W}\right)dn = \\ =(1+q)D\left(\frac{\sigma_a}{\sqrt{2}}\right)^q\int_{n_*}^{\bar{n}}\left[\frac{l(n)}{l(\bar{n})+\lambda(l(\bar{n}))-l(n)}\right]^{\frac{q}{2}}\cdot f\left(\frac{h}{W},\frac{l\left(n\right)}{W}\right)dn, \end{gathered} \tag{19}$$

where according to the cyclic plastic zone can be calculated by the formula [2]

$$\lambda(l\left(n\right)) = \frac{1}{8}\left(\frac{\pi\sigma_a\cdot f\left(\frac{h}{W},\frac{l(n)}{W}\right)}{2\sigma_\Upsilon}\right)^2 l\left(n\right). \tag{20}$$

Using Laplas transform to solve the equation (19) we obtain the set of constitutive equations

$$\left\{\begin{array}{l} \dfrac{dl}{dn} = D\left(1+\dfrac{1}{q}\right)\dfrac{1}{\left[2\lambda(l\left(n\right))\right]^{\frac{q}{2}-1}}\cdot\left(\sigma_a\sqrt{l}\cdot f\left(\dfrac{h}{W},\dfrac{l\left(n\right)}{W}\right)\right)^q \\ n_* = \dfrac{1}{(1+q)D}\left[\dfrac{1}{\sigma_a}\right]^q\left[\dfrac{2\lambda(l_0)}{l_0}\right]^{\frac{q}{2}}\cdot f\left(\dfrac{h}{W},\dfrac{l_0}{W}\right)^{-q} \end{array}\right., \tag{21}$$

where the first equation gives the fatigue crack propagation stage and the second one gives the duration of the incubation stage.

Models of the fatigue crack growth (21) involve material constants are to be determined from two base experiments. The first group characterizes the plastic behaviour of a material and determined from stress-strain diagrams in uniaxial tension tests.

The second group involves q and D coefficients and characterizes the material resistance to the accumulation of scattered fatigue macrodamage. The values of q and D are calculated by results of standard fatigue tests of plain cylindrical specimens under uniaxial reversed tension-compression presented in a form of the Wochler curve. In this case, the homogeneous stress state is realised in a specimen, fatigue macrodamage is accumulated uniformly through the whole of the specimen volume, and the completion of the incubation stage coincides practically with the full fracture of a specimen. So, on integrating of equation (6), for the conditions that the damage parameter $\omega(x, n)$ varies from 0 to 1.0 as the time n varies from zero to the number of cycles of fracture n_R , one obtains the following expression

$$n_R = \frac{1}{(1+q)D(\sigma_a)^q}. \tag{22}$$

Equation (22) is a straight line in a log-log plot of the stress amplitude σ_a against the number of cycles of fracture n_R . Specifically, the value of the exponent q is equal to the inverse slope of the straight line. On the whole, the values of q and D coefficients are calculated using an optimization method such as that due to Gauss-Newton to minimize the sum of squares of the residuals, with respect to the coefficients q and $(1+q)\,D$.

4 Example

Thus, in what follows the study of some features of fatigue crack growth in thin finite plates made of materials with different yield strength σ_Y values may be of interest. The materials investigated are aluminum alloys 2024-T3 and 7075-T6. The calculations are carried out for original crack length of $l_0 = 2,54mm$ in the plate with width $W = 305mm$ and height $h = 889mm$ under symmetrical high-cycle loading. Solving the problem of fatigue fracture based on fatigue crack growth model (21) requires information about value σ_Y , coefficients D and q as well as concretization of expression for function $f\left({}^{l}\!/_{W},\ {}^{h}\!/_{W}\right)$.

The values of stress-strain characteristics including the yield strength σ_Y as well as the values of q and D coefficients for the materials investigated are determined by experiments from [8] are given in the Table 1.

Table 10: Stress-strain and fatigue damage characteristics of investigated materials

alloy	σ_Y,MPa	$D, (\text{MPa}^q \cdot \text{cycle})^{-1}$	q
2024-T3	353	$7.45 \cdot 10^{-26}$	8.28
7075-T6	523	$3.33 \cdot 10^{-29}$	9.23

The function $f\left({}^{l}\!/_{W},\ {}^{h}\!/_{W}\right)$ is specified on the basis of polynomial approximation of numerical calculated results of the stress field obtained by the boundary collocation method [9]. For given finite plates with central crack analytical expression of correcting function is presented in a form

$$f\left({}^{l}\!/_{W},\ {}^{h}\!/_{W}\right) = 2,0833\left(\frac{l}{W}\right)^3 - 0,9536\left(\frac{l}{W}\right)^2 + 0,3781\left(\frac{l}{W}\right) + 0,9741. \tag{23}$$

The problem is reducing to integration fatigue crack growth rate equation in set (21) taking into account (23).

The results calculated on the basis of developed model for plates of 2024-T3 and 7075-T6 aluminum alloys are presented in Figure 4 in graphical form (solid lines) for crack length l as functions of number cycles n . Completely reversed uniaxial loading with different stress level is considered.

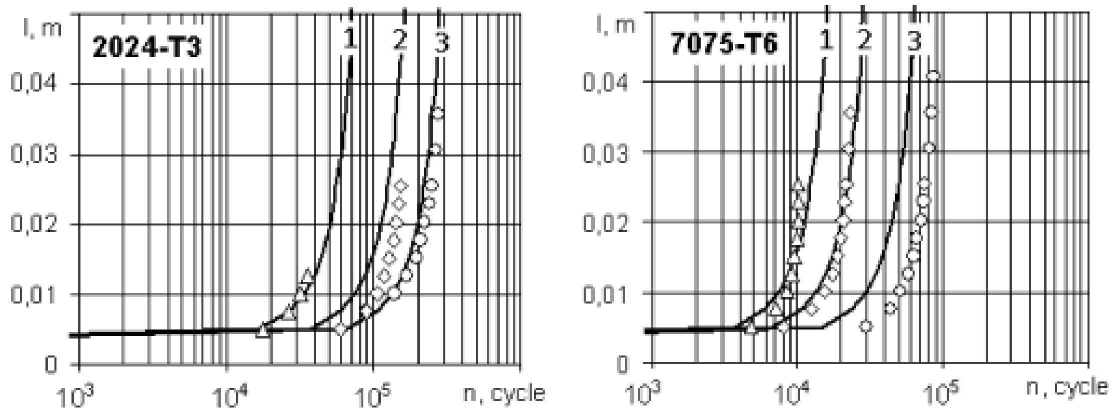

Figure 4: Dependence of crack length on the number of load cycles at $R = -1$ in aluminum alloys 2024-T3 ($\sigma_a = 52$ (3-$\circ$), 69 (2-$\diamond$), 103 (1-Δ) MPa) and 7075-T6 ($\sigma_a = 69$ (3-$\circ$), 103 (2-$\diamond$), 138 (1-Δ) MPa).

The experimental data taken from literature [10] are shown in the figure as well. The satisfactory agreement between the calculated results and experimental data confirms the adequacy of the constructed model.

5 Conclusion

The paper presents the theoretical approach to construct a phenomenological two-stage model of the fatigue crack growth, based on joint consideration of concepts of fracture mechanics and continuum damage mechanics. The received set of constitutive equation of the model includes the equations for incubation stage duration and crack propagation stage for fatigue crack in thin finite plate under completely reversed uniaxial loading. The modified method of boundary collocation and polynomial approximation are applied to receive analytical expression for stress distribution along fracture front. Use of correction function allows to consider the kinetic problem of fatigue crack growth for finite plates with different crack geometry. The material constants and coefficients used in constitutive set of model equations determined also the properties of materials and independent on crack geometry because they are calculated by the results of standard tests of plain cylindrical specimens.

The calculation results within the framework of the model for two aluminum alloys 2024-T3 and 7075-T6 agree well with experimental data.

References

[1] Golub V.P., Plashchynska A.V. A phenomenological model of fatigue crack growth in perfectly plastic infinite plates under completely reversed uni-axial loading . *Int. Appl. Mech.* 41, 12 (2005) 1426-1436

[2] Newman J. C., Jr. A crack-closure model for predicting fatigue crack growth under aircraft spectrum loading. *NASA TM - 81941*, (1981) 63

[3] Dubensky R.G. Fatigue crack propagation in 2024-T3 and 7075-T6 aluminum alloys at high stresses. *NASA CR-1732*, (1971) 32

[4] Newman J.C. An improved method of collocation for the stress analysis of cracked plate with various shaped boundaries. *NASA TN D-6376*, (1971) 45

[5] Muskhelishvili N.I. (J.R.M. Radok, transt). *Some basic problems of the mathematical theory of elasticity.* Third ed., P. Noordnoff, Ltd (Groningen), (1953).

[6] Hudson C.M., Scardina J.T. Effect of stress ratio on fatigue-crack growth in 7075-T6 aluminum-alloy sheet. *NASATMX- 60125*, (1967) 24

[7] Bolotin V.V. Model of fatigue crack with a tip region. *Soviet Applied Mechanics*, June (1988) 1154-1159.

[8] Grover H. J., Hyler W. S., Kyhn P., Landers C. B. and Howell F. M.Axial-Load Fatigue Properties of 24S-T and 75S-T Aluminum Alloy as Determined in Several Laboratories. *NASA TN-2928,* (1953) 64.

[9] Murakami Y. (Editor-in-chief) *Stress intensity factors handbook.* Volume 1. Pergamon Press. NY. USA (1987)

[10] Hudson C.M. Effect of stress ratio on fatigue crack growth in 7075-T6 and 2024-T3 aluminum alloy specimens. *NASA TN D 5390*, (1969) 34

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