# A strain-softening bar revisited

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**Abstract**

Dynamic processes in an elastic bar with strain-softening are considered from the standpoint of the modern theory of phase transitions. Namely, the possibility of a new phase nucleation as a result of two shock waves collision is investigated. The model of an elastic body with non-convex strain energy is used. The stress is assumed to be a piecewise linear function of strain, containing a negative slope. The different particular cases depending on the relations between elastic modules of each part of the stress-strain curve are analytically studied. The obtained solutions are compared with the classical ones derived by Z.P. Bažant et al. [2–4]

## 1 Introduction

In the paper we consider the problem concerned with a new phase nucleation due to two shock waves collision in an elastic bar made of a material, capable of undergoing phase transitions. The model of an elastic body with non-convex strain energy potential [5–15] is used. In the present study the stress is assumed to be a piecewise linear function of strain. It is known that the problem of elastostatics for a material with non-convex strain energy can have solutions with discontinuous deformation gradients [1]. In the framework of the model, the surfaces of the strain discontinuity are considered as the phase boundaries, and the domains of continuity are considered as zones, which are occupied by different phases of the material. The solution of both statical and dynamical problems are generally non-unique, therefore an additional thermodynamic boundary condition [5–11] at the phase boundary is required.

In the study below we present the mathematical problem formulation in the framework of the model described above and its analytical and numerical investigation. We consider analytically several important particular limiting cases. The analytical solutions are compared with the numerical ones. The classical study concerned with the shock waves collision in the bar with strain-softening is given in [2–4] by Bažant, Belytschko et al. In section 7 we compare the solutions derived in [2–4] and the results obtained in the present paper from the standpoint of the modern theory of phase transitions.

## 2 The problem formulation

Consider a direct elastic bar with a cross-sectional area $S$ (see Fig. 1). Let $u(x,t)$ be the longitudinal displacements of the bar cross-sections. Here $x$ $(-L < x < L)$ is the position of a cross-section, $t$ is time, $2L$ is the length of the bar. Denote by prime and dot the derivative with respect to $x$ and $t$, respectively. Strain $\varepsilon = u'$ is assumed to be infinitesimal. We consider the pure mechanical theory, hence, any thermal effects are neglected. The material of the bar is capable of undergoing phase transitions, therefore the

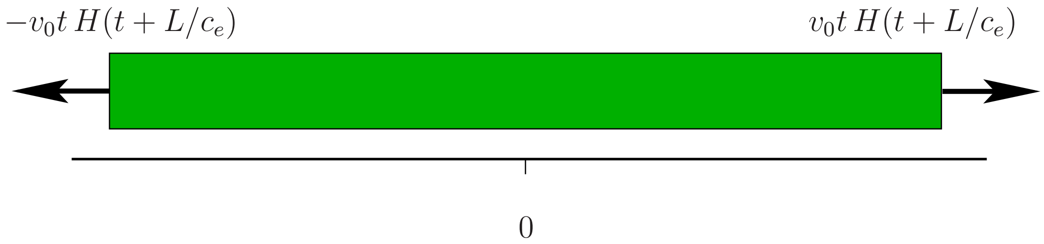


Figure 1: Schematic of the system

stress-strain relation have to be non-monotonic, i.e. the elastic potential for the material must be non-convex. We assume the stress $\sigma$ to be a piecewise linear function of strain $\varepsilon$ (Fig. 2):

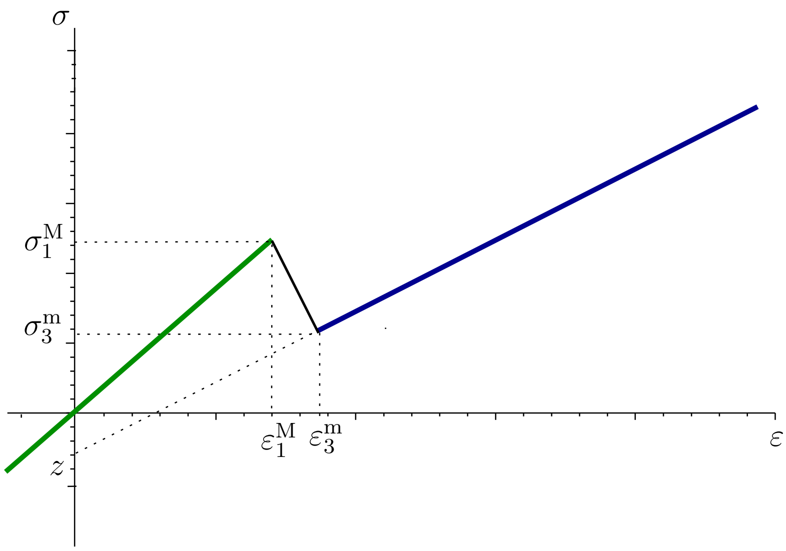


Figure 2: Stress-strain curve

$$\sigma(\varepsilon) = \begin{cases} E_1\varepsilon, & 0 < \varepsilon \leq \varepsilon_1^M; \\ -E_2\varepsilon + \varepsilon_1^M (E_1 + E_2), & \varepsilon_1^M < \varepsilon \leq \varepsilon_3^m; \\ E_3\varepsilon + z, & \varepsilon > \varepsilon_3^m. \end{cases} \tag{1}$$

$$z = -\varepsilon_3^m (E_2 + E_3) + \varepsilon_1^M (E_1 + E_2). \tag{2}$$

Here $E_i, i = \overline{1,3}$ are the elastic moduli for corresponding intervals of stress-strain curve, symbols $\varepsilon_1^{\mathrm{M}}$ and $\varepsilon_3^{\mathrm{m}}$ are defined in Fig. 2. Intervals $0 < \varepsilon \leq \varepsilon_1^M$ and $\varepsilon > \varepsilon_3^m$ correspond to the stable states of the material (phases 1 and 3, respectively). Phase 2 ($\varepsilon_1^M < \varepsilon \leq \varepsilon_3^m$), which correspond to the incident part of the strain-stress relation, is unstable.

The governing equation for the bar has the form:

$$S\sigma' - \rho\ddot{u} = 0. \tag{3}$$

Here $\rho$ is the mass density for the material of the bar, which is assumed to be the same for the both stable phases of the material. Since $\sigma' = E_i u''$, the governing equation is

$$u'' - c^2\ddot{u} = 0. \tag{4}$$

Here $c = c_e = \sqrt{\frac{E_1}{\rho}}$ if the material of the bar is in the phase state 1, and $c = c_i = \sqrt{\frac{E_3}{\rho}}$ if the material of the bar is in the phase state 3.

Consider the bar excited with two kinematic loadings:

$$u_e^-(-L) = -v_0(t + L/c_e)H(t + L/c_e), \tag{5}$$

$$u_e^+(L) = v_0(t + L/c_e)H(t + L/c_e), \tag{6}$$

where rate of loading $v_0 > 0$ is supposed to be a subcritical one, namely, $v_0 < c_e$. At the initial moment of time the displacements and velocities of the bar cross-sections are equal to zero:

$$u = 0, \tag{7}$$

$$\dot{u} = 0. \tag{8}$$

For definiteness we suppose that initially the material of the bar was in phase state 1. We assume that strain $\varepsilon_1^M$ is such that $v_0/c_e < \varepsilon_1^M < 2v_0/c_e$. Hence, for times $-L/c_e < t < 0$ the problem solution is the solution of Eq. (4) together with initial conditions (7)–(8) and boundary conditions (5)–(6), namely, this is a superposition of two shock waves. Therefore, the strain is as follows:

$$\varepsilon = v_0/c_e \left(H(t - x/c_e) + H(t + x/c_e)\right). \tag{9}$$

At the moment of time $t = 0$ the two shock waves collides. Due to the system symmetry with respect to the middle of the bar, the collision of the shock waves happens at $x = 0$. In the linearly elastic bar such a collision will generate an infinitesimal area in a vicinity of $x = 0$ with strain $2v_0/c_e > \varepsilon_1^M$. One can see that for the material with constitutive equation (1) the collision leads to a new phase nucleation in a small vicinity of $x = 0$. In what follows we consider only the case when the new phase is stable, namely, the material of the bar is in the phase state 3 in a small neighborhood of $x = 0$. Let us also consider only symmetrical solutions with two phase boundaries with positions $x = \pm l(t)$ (see Fig. 3), respectively.

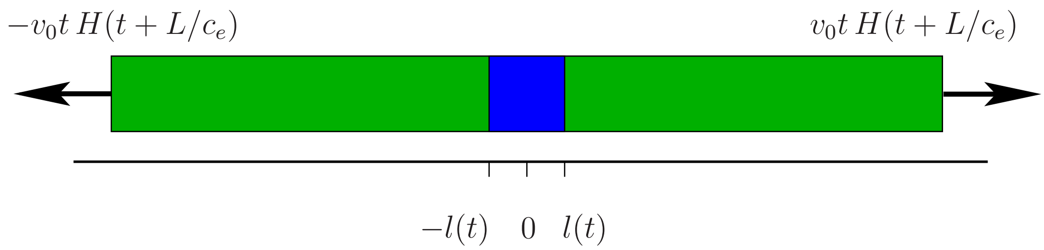


Figure 3: The new phase nucleation

In what follows, we adopt the notation: $[\mu] = \mu_e - \mu_i$, $\langle\mu\rangle = (\mu_e + \mu_i)/2$, $\mu_e = \mu|_{x=l(t)+0}$,$\mu_i = \mu|_{x=l(t)-0}$ for any arbitrary function $\mu(x, t)$. If the phase boundary exists at $x = l$, the following boundary conditions have to be satisfied:

$$[u] = 0, \tag{10}$$

$$[\sigma] = -\rho\dot{l}[\dot{u}]. \tag{11}$$

Equations (10)–(11) are the Hugoniot conditions for Eq. (3). These two conditions are the continuity condition and the equation for balance of momentum for the infinitesimal interval $[l(t) - 0; l(t) + 0]$, respectively. Differentiating of Eq. (10) with respect to $t$ yields

$$[\dot{u} + \dot{l}u'] = 0. \tag{12}$$

Thus, Eq. (11) can be rewritten in the form

$$[\sigma] = \rho \dot{l}^2 [\varepsilon]. \tag{13}$$

In what follows for simplification we will put the density $\rho = 1$.

To find the unknown function $l(t)$ one needs to formulate an additional constitutive equation (so-called thermodynamical boundary condition). It can be shown (see e.g. ([6, 11]), that the second law of thermodynamics localized to the infinitesimal interval $[l(t) - 0; l(t) + 0]$ containing the phase boundary leads to the following inequality:

$$-\mathcal{F}\dot{l} \geq 0, \tag{14}$$

where

$$\mathcal{F} = -S([W] - \langle\sigma\rangle[\varepsilon]) \tag{15}$$

is the configurational (material, thermodynamical) force [11, 16, 17] on the phase boundary,

$$SW = S\int \sigma(\varepsilon)\, d\varepsilon \tag{16}$$

is the strain energy per unit length of the bar. This inequality follows from full dynamical consideration, which involves the inertia forces. The possible point of view is that inequality (14) is important, and perhaps unique, restriction on the constitutive equation structure. The following kinds of the additional thermodynamical condition are most widespread in the literature:

$$\mathcal{F} = 0; \tag{17}$$

$$\dot{l} = -\gamma^{-1}\mathcal{F}. \tag{18}$$

Here $\gamma > 0$ is a material constant associated with dissipation on the phase boundary. We will call relations (17) and (18) *the non-dissipative condition* and *the simplest dissipative condition*, respectively. It may be noted that Eq. (17) is the limit case for Eq. (18) as $\gamma \to +0$.

## 3 The strains at the phase boundary

The symmetry of the mechanical system allows one to consider only the left part of the bar $-L < x < 0$ with the unique phase boundary $l(t) < 0$. Let us call the part $-L < x < l(t)$ of the bar, where the material is in phase state 1, "the outer area", and the part $l(t) < x < 0$ of the bar, where the material is in phase state 3, "the inner area".

### 3.1 The displacements in the outer area of the bar

Let us represent the displacements of cross-sections of the bar in outer area $u_e$ in the form:

$$u_e = u_e^+ + u_e^-, \tag{19}$$

where $u_e^+$ is the incidental wave, and $u_e^-$ is the wave, reflected from the phase boundary.

The expression for the incidental wave $u_e^-$ must satisfy wave equation (4), initial conditions (7)–(8), and boundary condition (6). Hence, for $u_e^-$ one can write down:

$$u_e^- = v_0\Big(t - \frac{x}{c_e}\Big)H\Big(t + \frac{x}{c_e}\Big). \tag{20}$$

In what follows we consider only times $0 < t < L/c_e$. Hence, the expression for the incidental wave is:

$$u_e^- = \frac{v_0}{c_e}(x - c_e t). \tag{21}$$

For the reflected wave $u_e^+$ one can write down the expansion:

$$u_e^+ = u_{e1}^-(x + c_e t) + o(x), \tag{22}$$

where coefficient $u_{e1}$ will be defined below from the conditions at the phase boundary.

### 3.2 The displacements in the inner area of the bar

Represent the displacements $u_i$ in the inner area of the bar in the form:

$$u_i = u_{i1}^-(x - c_i t) + u_{i1}^+(x + c_i t) + o(x). \tag{23}$$

Denote strain $\varepsilon_i$ in the inner part of the bar at the point $x = 0$, $t = 0$ as follows:

$$\varepsilon_i = 2\varepsilon_0. \tag{24}$$

Due to the symmetry of the problem $\dot{u}_i = 0$ at $x = 0$, $t = 0$. Hence,

$$u_{i1}^- = u_{i1}^+ = \varepsilon_0. \tag{25}$$

### 3.3 Expressions for the strains at the phase boundary

Let us obtain the expressions for strain $\varepsilon$, speed $v$ and stress $\sigma$ both in inner and outer areas of the bar.

Differentiating Eq. (19) for displacements $u_e^-$ with respect to $x$ and $t$, taking into account that $u_e^-$ is given by Eq. (21), and $u_e^+$ is given by Eq. (22), one gets for strain $\varepsilon_e$ and speed $v_e$ in the outer area of the bar:

$$\varepsilon_e = \frac{v_0}{c_e} + u_{e1}^-, \tag{26}$$

$$v_e = -v_0 + c_e u_{e1}^-, \tag{27}$$

According to Eq. (1) we can write down for stress $\sigma_e$ in the outer area:

$$\sigma_e = E_1\varepsilon_e, \tag{28}$$

From Eqs. (23),(25) one can see that speed in the inner area $v_i = 0$. In accordance with Eq. (1), for stress $\sigma_i$ one has:

$$\sigma_i = E_3\varepsilon_i + z. \tag{29}$$

Substituting Eqs. (26)–(29) into Hugoniot conditions (12),(13), we obtain the set of linear algebraic equations with unknown $u_{e1}^-$ and $\varepsilon_i$:

$$\begin{aligned} &-\varepsilon_i\dot{l} + (c_e + \dot{l})u_{e1}^- = \frac{v_0}{c_e}(c_e - \dot{l}),\\ &\varepsilon_i(c_i^2 - \dot{l}^2) - (c_e^2 - \dot{l}^2)u_{e1}^- = \frac{v_0}{c_e}(c_e^2 - \dot{l}^2) - z. \end{aligned} \tag{30}$$

Solving this set of equations we find out the expressions for amplitudes of strains at the phase boundary:

$$\varepsilon_i = \frac{2v_0(c_e - \dot{l}) - z}{-\dot{l}c_e + c_i^2}, \tag{31}$$

$$\varepsilon_e = \frac{v_0}{c_e} + u_{e1}^- = \frac{2v_0(c_i^2 - \dot{l}^2) - \dot{l}z}{(c_e + \dot{l})(-\dot{l}c_e + c_i^2)}. \tag{32}$$

### 3.4 Necessary conditions for the new phase nucleation

It is clear (see Fig. 2)) that to observe a new phase nucleation one should satisfy the following conditions:

$$\varepsilon_e < \varepsilon_1^M, \tag{33}$$

$$\varepsilon_i > \varepsilon_3^m. \tag{34}$$

The first inequality guarantees that after the shock waves collision the material in the outer area of the bar remains in phase state 1, and its behaviour can be described by means of the following constitutive equation:

$$\sigma = E_1\varepsilon. \tag{35}$$

The second condition means that the material of the bar in the inner area has transformed to phase state 3, and its behaviour is given by the constitutive equation:

$$\sigma = E_3\varepsilon + z. \tag{36}$$

Let us also take into account the two following restrictions. Firstly, one should formulate the condition which garantees that the material of the whole bar could not remain in the phase state 1 after the shock waves collision:

$$2\frac{v_0}{c_e} > \varepsilon_1^M. \tag{37}$$

In other words this condition means that the material is not a linearly elastic one.

Secondly, one can write down an inequality which means that before the waves collision the material of the whole rod was in phase state 1:

$$\frac{v_0}{c_e} < \varepsilon_1^M. \tag{38}$$

## 4 The expression for the configurational force

Configurational force is given by Eq. (15). For the jump of the strain energy the following formula can be written:

$$[W] = \int_{\varepsilon_1}^{\varepsilon_1^M} \sigma(\varepsilon) d\varepsilon + \int_{\varepsilon_1^M}^{\varepsilon_3^m} \sigma(\varepsilon) d\varepsilon + \int_{\varepsilon_3^m}^{\varepsilon_3} \sigma(\varepsilon) d\varepsilon. \tag{39}$$

According to Eq. (1), for $[W]$ we can write down:

$$[W] = -\frac{1}{2} E_1 \varepsilon_e^2 + \frac{1}{2} E_3 \varepsilon_i^2 + z\varepsilon_i + \nu, \tag{40}$$

where

$$\nu = E_1 \frac{(\varepsilon_1^M)^2}{2} + \int_{\varepsilon_1^M}^{\varepsilon_3^m} \sigma(\varepsilon) d\varepsilon - \left( E_3 \frac{(\varepsilon_3^m)^2}{2} + z\varepsilon_3^m \right) \tag{41}$$

or after integration one obtains:

$$\nu = -\frac{1}{2}(E_1 + E_2)(\varepsilon_1^M)^2 + \frac{1}{2}(E_2 + E_3)(\varepsilon_3^m)^2 = \text{const}. \tag{42}$$

For term $\langle\sigma\rangle[\varepsilon]$ in Eq. (15) one has:

$$\langle\sigma\rangle[\varepsilon] = -\frac{1}{2} E_1 \varepsilon_e^2 + \frac{1}{2} E_3 \varepsilon_i^2 - \frac{1}{2}(E_3 - E_1)\varepsilon_e \varepsilon_i - \frac{1}{2} z(\varepsilon_e - \varepsilon_i). \tag{43}$$

Thus, the expression for configurational force $\mathcal{F}$ is:

$$\mathcal{F} = -S\Big(\frac{1}{2}(E_3 - E_1)\varepsilon_e \varepsilon_i + \frac{1}{2} z(\varepsilon_i + \varepsilon_e) + \nu\Big). \tag{44}$$

Without loss of generality, we assume that the cross-sectional area of the bar $S = 1$.

By substituting Eq. (44) together with Eqs. (31),(32) into the additional thermodynamical condition in the form of Eq. (17) or Eq. (18), one obtains an algebraic equation of the 3 of 4 order, respectively, for the phase boundary speed $\dot{l}$. Below we will consider mostly the non-dissipative condition (17), for which one obtains:

$$\begin{aligned}(c_i^2 - c_e^2) \frac{2v_0(c_e - \dot{l}) - z}{-\dot{l}c_e + c_i^2} \cdot \frac{2v_0(c_i^2 - \dot{l}^2) - \dot{l}z}{(c_e + \dot{l})(-\dot{l}c_e + c_i^2)}& \\ + z\left( \frac{2v_0(c_e - \dot{l}) - z}{-\dot{l}c_e + c_i^2} + \frac{2v_0(c_i^2 - \dot{l}^2) - \dot{l}z}{(c_e + \dot{l})(-\dot{l}c_e + c_i^2)} \right) + 2\nu = 0.&\end{aligned} \tag{45}$$

Eq. (45) can be transformed to the algebraic equation of the 3 order for phase boundary speed $\dot{l}$ ($\dot{l} \neq -c_e$).

Eq. (45) can be solved numerically. The results of numerical simulation shows that the solutions satisfying restrictions (33), (34) exist. At the same time the loadings under consideration satisfy inequalities (37),(38). Thus, the scenario of a new phase nucleation, proposed in this study, can be realised.

By means of system for analytical simulations Maple 12 the analytical solution of Eq. (45) is derived. However, the solution is rather complex, with multiple parameters and non-informative. Therefore, below we will consider some important particular cases, whose investigation leads to clear results and better understanding of the mechanical system behaviour.

# 5 Some particular cases. The analytical solutions.

## 5.1 The phase boundary speed is a near-critical one

Consider the case when phase boundary speed $\dot{l}$ tends to its critical value $-c_e + 0$. For $\dot{l} \to -c_e + 0$ from Eq. (30) one has:

$$\varepsilon_i = -\frac{z}{c_i^2 - c_e^2}. \tag{46}$$

One can see that for this limiting case the set of equations (30) is solvable with respect to $\varepsilon_i$ only for the rate of loading

$$v_0 = v_{0\mathrm{cr}} = -\frac{c_e z}{2(c_i^2 - c_e^2)}. \tag{47}$$

If $c_i > c_e$, namely, when the stiffness of the inner phase is greater than the stiffness of the outer phase, magnitude $z$ is always negative. Hence, there always exist certain value $v_0 = v_{0\mathrm{cr}} > 0$, given by Eq. (47), such that phase boundary speed $\dot{l}$ tends to its critical value.

If $c_i < c_e$, i.e.the inner phase is softer than the outer one, $z > 0$ when $c_i < \sqrt{E_{3*}}$ and $z < 0$ when $c_i > \sqrt{E_{3*}}$, where

$$E_{3*} = -E_2 + \frac{\varepsilon_1^M}{\varepsilon_3^m}(E_1 + E_2). \tag{48}$$

Thereby, according to Eq. (47) critical value of rate of loading $v_{0\mathrm{cr}} > 0$ only if $z > 0$, i.g., $c_i < \sqrt{E_{3*}}$. Otherwise, phase boundary speed $\dot{l}$ can not be closed to its critical value $c_e$.

Recall that rate of loading $v_0 = v_{0\mathrm{cr}}$ must satisfy inequalities (37), (38).

It can be shown that $\dot{l} \to c_e + 0$ for $v_0$, given by Eq. (47), is a root of non-dissipative thermodynamical condition (17).

## 5.2 Stiffness of the inner part $c_i \to 0$

Consider the situation when the stiffness of an inner phase $c_i \to 0$. In what follows we assume that $\sigma_i(\varepsilon_3^m) = 0$.

For $E_3 \to 0$ ($c_i \to 0$), $\sigma_i(\varepsilon_3^m) = 0$ according to Eq. (1) one has:

$$z = -c_i^2 \varepsilon_3^m \to 0 \tag{49}$$

We assume that $\dot{l}$ has a finite value. Represent:

$$c_i^2 = A_1 \mu, \tag{50}$$

$$\dot{l} = \dot{l}_0 + B_1 \mu + o(\mu), \tag{51}$$

where $\mu$ is a formal small parameter. From Eq. (41) one can derive:

$$\nu = \nu_0 + O(\mu). \tag{52}$$

Here

$$\nu_0 = E_1 \frac{(\varepsilon_1^M)^2}{2} + \int_{\varepsilon_1^M}^{\varepsilon_3^m} \sigma(\varepsilon) d\varepsilon > 0. \tag{53}$$

Hence, from Eq. (45) together with expansions (50)–(51) one obtains:

$$\frac{\dot{l}_0(-2v_0^2+\nu_0)+(2v_0^2+\nu_0)c_e}{c_e+\dot{l}_0}=0. \tag{54}$$

Solving the above equation we find out that

$$\dot{l}=\dot{l}_0+O(\mu)=\frac{2v_0^2+\nu_0}{2v_0^2-\nu_0}c_e+O(\mu). \tag{55}$$

Eq. (55) shows that for $\nu_0>0$ phase boundary speed $\dot{l}<0$ if $|v_0|<\sqrt{\frac{\nu_0}{2}}$. However, at the same time $|\dot{l}|>c_e$, i.e. the phase boundary speed is a supercritical one, while we are looking for the subcritical solutions only.

Let us assume that phase boundary speed $\dot{l}$ is a small magnitude, and that the magnitudes $\dot{l}^2$, $c_i^2$, and $z$ are of the same asymptotic order. So, for $\dot{l}$ we can write down an expansion:

$$\dot{l}=\dot{l}_1\mu^{1/2}+o(\mu^{1/2}). \tag{56}$$

Then from Eq. (45) we obtain:

$$\frac{\dot{l}_1^{\,2}(2v_0^2+\nu_0)-2v_0^2A_1}{(-\dot{l}_1c_e+A_1)^2}=0. \tag{57}$$

One can see that

$$\dot{l}=\pm\sqrt{\frac{2v_0^2c_i^2}{2v_0^2+\nu_0}}+o(c_i^2). \tag{58}$$

Since we consider phase boundary speed $\dot{l}<0$, we choose the following solution:

$$\dot{l}=-\sqrt{\frac{2v_0^2c_i^2}{2v_0^2+\nu_0}}+o(c_i^2). \tag{59}$$

Eq. (59) shows that for $c_i\to 0$ phase boundary speed $\dot{l}\to 0$.

So, we have found out 3 roots, given by Eqs. (55),(58), of thermodynamical equation (45) We have discussed above that Eq. (45) for $\dot{l}\neq -c_e$ can be transformed to a cubic one. Thus, we have obtained all possible solutions of Eq. (45) for asymptotically small magnitudes $c_i^2$ and $z$.

Eq. (59) is validated by computer simulations carried out by means of Maple 12. For example, for the parameters:

$$E_1=1,\quad E_2=1.5,\quad E_3=10^{-4},\quad \varepsilon_1^M=0.3,\quad \varepsilon_3^m=0.5,\quad v_0=0.18 \tag{60}$$

the numerical solution of Eq. (45) gives us the following value for phase boundary speed:

$$\dot{l}=-0.00678. \tag{61}$$

At the same time according to Eq. (59)

$$\dot{l}=-0.00680. \tag{62}$$

By substituting Eq. (59) into Eqs. (31),(32) we obtain the expressions for strains at the phase boundary:

$$\varepsilon_i=\frac{2\sqrt{v_0^2+\nu_0/2}}{c_i}+O(1),\qquad \varepsilon_e=\frac{c_i\nu_0}{c_e^2\sqrt{v_0^2+\nu_0/2}}+O(c_i^2). \tag{63}$$

### 5.3 Stiffness of the inner phase $c_i \to \infty$

Consider the case when the stiffness of the inner phase $c_i \to \infty$. Represent $c_i = A_1\mu^{-1}$, $\dot{l} = \dot{l}_0 + O(\mu)$, where $\mu$ is a formal small parameter, $A_1$ is a constant. Substitute these expansions into Eq. (45) and balance the corresponding terms of the same orders. It is easy to demonstrate that coefficient before $\mu^{-1}$ equals to zero. For the coefficient before $\mu^0$ we get:

$$\frac{2v_0 - \varepsilon_3^m c_e}{c_e + \dot{l}_0}((2v_0 c_e - a) - \dot{l}_0(2v_0 - c_e\varepsilon_3^m)) \\ + (a - c_e^2\varepsilon_3^m)\frac{2v_0 + \dot{l}_0\varepsilon_3^m}{c_e + \dot{l}_0} + a\varepsilon_3^m + 2b = 0. \quad (64)$$

Here we denote:

$$a = z + \varepsilon_3^m c_i^2 = -\varepsilon_3^m E_2 + \varepsilon_1^M(E_1 + E_2), \quad (65)$$

$$b = \nu - \frac{1}{2}(\varepsilon_3^m)^2 c_i^2 = -\frac{1}{2}(E_1 + E_2)(\varepsilon_1^M)^2 + \frac{1}{2}E_2(\varepsilon_3^m)^2. \quad (66)$$

From Eq. (64) one obtains for $\dot{l}_0$:

$$\dot{l}_0 = \frac{c_e(2v_0^2 - 2v_0 c_e\varepsilon_3^m + a\varepsilon_3^m + b)}{2v_0^2 - a\varepsilon_3^m - b - 2v_0 c_e\varepsilon_3^m + c_e^2(\varepsilon_3^m)^2}. \quad (67)$$

Then for phase boundary speed $\dot{l}$ we can write down:

$$\dot{l} = \frac{c_e(2v_0^2 - 2v_0 c_e\varepsilon_3^m + a\varepsilon_3^m + b)}{2v_0^2 - a\varepsilon_3^m - b - 2v_0 c_e\varepsilon_3^m + c_e^2(\varepsilon_3^m)^2} + O(\mu). \quad (68)$$

Eq. (68) is validated by computer simulations carried out by means of Maple 12. For example, for the parameters:

$$E_1 = 1, \quad E_2 = 1.5, \quad E_3 = 1000, \quad \varepsilon_1^M = 0.3, \quad \varepsilon_3^m = 0.4, \quad v_0 = 0.18 \quad (69)$$

the numerical solution of Eq. (45) gives us the following value for phase boundary speed:

$$\dot{l} = -0.8790. \quad (70)$$

At the same time according to Eq. (68)

$$\dot{l} = -0.8790. \quad (71)$$

### 5.4 Inner and outer areas are of the same stiffnesses ($c_i = c_e$)

Here we consider the situation when the stiffnesses of inner and outer areas are equal to each other, e.g. $c_i = c_e$. In this case Eqs. (31),(32) for strains at the phase boundary can be transformed to the following form:

$$\varepsilon_i = \frac{2v_0(c_e - \dot{l}) - z}{-c_e(\dot{l} - c_e)}, \quad (72)$$

$$\varepsilon_e = \frac{2v_0(c_e^2 - \dot{l}^2) - \dot{l}z}{-c_e(\dot{l} - c_e)(\dot{l} + c_e)}. \quad (73)$$

Thermodynamical condition (17), where the configurational force is given by Eq. (44), can be written as follows:

$$(\varepsilon_i + \varepsilon_e)z + 2\nu = 0. \tag{74}$$

Substitution of Eqs. (72),(73) into Eq. (74) leads to quadratic equation for phase boundary speed $\dot{l} \neq -c_e$:

$$(4v_0 z + 2\nu c_e)\dot{l}^2 + 2z^2\dot{l} - ((4v_0 z + 2\nu c_e)c_e - z^2)c_e = 0. \tag{75}$$

The solution of Eq. (75) has the form:

$$\dot{l} = \frac{-2z^2 \pm \sqrt{D}}{2(4v_0 z + 2\nu c_e)}, \tag{76}$$

where discriminant $D$ is given by the expression:

$$D = 4z^4 + 4(4v_0 z + 2\nu c_e)^2 c_e^2 - 4(4v_0 z + 2\nu c_e)z^2 c_e. \tag{77}$$

Denote $\gamma = \dfrac{c_e}{z^2}(4v_0 z + 2\nu c_e)$. Represent discriminant $D$ in the form:

$$D = 4z^4(1 - \gamma + \gamma^2). \tag{78}$$

Investigating the quadratic function $1-\gamma+\gamma^2$ it is easy to demonstrate that $(1-\gamma+\gamma^2) > 0$ and its minimum value is $3/4$. Therefore,

$$D > 3z^4 > 0. \tag{79}$$

Hence, according to Eq. (76) the expression for phase boundary speed $\dot{l}$ is given by equation:

$$\dot{l} = \frac{-2z^2 + \sqrt{D}}{2(4v_0 z + 2\nu c_e)} < 0. \tag{80}$$

Substituting Eq. (77) into Eq. (80), we find out that phase boundary speed $\dot{l}$ in the case when stiffnesses of the inner and outer area are equal to each other, namely, $c_i = c_e$ is described by the following formula:

$$\dot{l} = \frac{-2z^2 + \sqrt{4z^4 + 4(4v_0 z + 2\nu c_e)^2 c_e^2 - 4(4v_0 z + 2\nu c_e)z^2 c_e}}{2(4v_0 z + 2\nu c_e)}. \tag{81}$$

Eq. (81) is validated by computer simulations carried out by means of Maple 12. For example, for the parameters:

$$E_1 = 1, \quad E_2 = 1.5, \quad E_3 = 1, \quad \varepsilon_1^M = 0.3, \quad \varepsilon_3^m = 0.4, \quad v_0 = 0.2 \tag{82}$$

the numerical solution of Eq. (45) gives us the following value for phase boundary speed:

$$\dot{l} = -0.622. \tag{83}$$

At the same time according to Eq. (81)

$$\dot{l} = -0.622. \tag{84}$$

## 6 Impossibility of transitions to phase state 2

Let us investigate the possibility of nucleation of a new phase corresponding to the incident part of the stress-strain curve. Namely, we study the possibility of phase transformation to phase state 2. The necessary condition for such phase transformation is thermodynamical condition given by Eq. (14), where the configurational force is given by Eq. (15). Recall that phase boundary speed $\dot{l} < 0$. Hence, in accordance with Eqs. (14),(15), the following inequality should be satisfied:

$$[W] - \langle\sigma\rangle[\varepsilon] \geq 0. \tag{85}$$

Let us denote strain in the outer area $\varepsilon_e = \varepsilon_1$ and strain in the inner area, e.g. the area where the material is in new phase state 2, $\varepsilon_i = \varepsilon_2$. The jump of energy $[W]$ equals to the area under the stress-strain curve for $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, term $\langle\sigma\rangle[\varepsilon]$ is equal to the area of trapezium, formed as a result of the intersections of the following straight lines:

$$\begin{aligned} &\sigma = 0, \quad \varepsilon = \varepsilon_1, \quad \varepsilon = \varepsilon_2, \\ &\sigma = \frac{\varepsilon_1^M(E_1+E_2) - E_1\varepsilon_1 - E_2\varepsilon_2}{\varepsilon_2 - \varepsilon_1}\varepsilon + \frac{(E_1+E_2)(\varepsilon_2 - \varepsilon_1^M)\varepsilon_1}{\varepsilon_2 - \varepsilon_1}. \end{aligned} \tag{86}$$

One can see that the area under the stress-strain curve restricted with the boundaries of the interval $\varepsilon \in [\varepsilon_1, \varepsilon_2]$ is always greater than the area of the mentioned above trapezium. Thus, the inequality (85) can not be satisfied, and therefore the thermodynamical condition given by Eq. (14) also cannot be fulfilled. Note that we have demonstrated the impossibility to satisfy the thermodynamical condition both in non-dissipative and dissipative formulation. Hence, the phase transition to phase state 2 in the mechanical system under consideration is impossible.

## 7 On the solution by Bažant et al.

In study [2–4] by Bažant et al. the authors consider the problem concerned with the shock waves collision in the strain-softening bar in the formulation similar to the one proposed in the present investigation. The authors of the cited works have obtained the solution of the problem in two particular cases, namely, in the case when stiffnesses of inner and outer areas are equal, e.g. $E_1 = E_3$, and the case when stiffness of the inner part $E_3 = 0$. Let us note that Bažant et al. in [2–4] do not use thermodynamical condition (14) as an additional constitutive equation which is necessary to formulate the fully defined problem. As an additional condition they use in [3, 4] the following assumption:

$$\varepsilon_e = \varepsilon_*, \tag{87}$$

$$\varepsilon_* = \frac{\sigma(\varepsilon_3^m)}{E_1} = \varepsilon_3^m + \frac{z}{E_1}, \tag{88}$$

since, as the authors declare, the numeric solution converges to $\varepsilon = \varepsilon_*$.

In what follows we will demonstrate that the solution obtained in the present study and derived with the use of thermodynamical condition does not fulfill Eq. (87).

Let us treat as an example the case of equal stiffnesses ($E_1 = E_3$). Consider the thermodynamical condition in the form of non-dissipative condition (17). Recall that the configurational force is given by Eq. (44). For the case of equal modules for phases 1 and 3, taking into account Eq. (87), one has:

$$z(\varepsilon_i + \varepsilon_*) + 2\nu = 0. \tag{89}$$

From the above equation one can easily derive:

$$\varepsilon_i = -\frac{2\nu}{z} - \varepsilon_*. \tag{90}$$

Substituting Eq. (90) into Hugoniot conditions (72),(73), written for the case when $E_1 = E_3$, and taking into account Eq. (87), we obtain the following set of equations for phase boundary speed $\dot{l}$:

$$\frac{2v_0(c_e - \dot{l}) - z}{-c_e(\dot{l} - c_e)} = -\frac{2\nu}{z} - \varepsilon_*, \tag{91}$$

$$\frac{2v_0(c_e^2 - \dot{l}^2) - \dot{l}z}{-c_e(\dot{l}^2 - c_e^2)} = \varepsilon_*. \tag{92}$$

One can see that the above set of equation is overdetermined. Eq. (87) and thermodynamic condition (17) lead to the same result only if Eq. (91) and Eq. (92) are identical. However, it is easy to demonstrate that these equations are identical only for

$$\dot{l} = -\sqrt{\frac{E_2 E_1}{E_2 + E_1}}. \tag{93}$$

At the same time Eq. (93) for phase boundary speed $\dot{l}$ does not coincide with the expression obtained for $\dot{l}$ in [3, 4]:

$$\dot{l} = -v_0(-A + \sqrt{1 + A^2}), \qquad A = \frac{c_e(\varepsilon_3^m - \varepsilon_*)}{2(2v_0 - c_e\varepsilon_*)}. \tag{94}$$

Actually, Eq. (94) depends on rate of loading $v_0$, and Eq. (93) does not.

Nevertheless, we can demonstrate that the solution obtained in study [3, 4] for the case when $E_1 = E_3$ fulfills the second low of thermodynamics in the form of Eq. (14). Actually, Eq. (14) together with Eq. (44) for the configurational force in the case when $E_1 = E_3$ leads to the following inequality:

$$z(\varepsilon_i + \varepsilon_e) + 2\nu \le 0. \tag{95}$$

Here we have also taken into account the fact that $\dot{l} < 0$. For $E_1 = E_3$ one can write down:

$$z = -(E_2 + E_1)(\varepsilon_3^m - \varepsilon_1^M), \qquad 2\nu = (E_2 + E_1)(\varepsilon_3^{m2} - \varepsilon_1^{M2}). \tag{96}$$

Hence we can rewrite inequality (95) as follows:

$$-z(-\varepsilon_i - \varepsilon_e + \varepsilon_3^m + \varepsilon_1^M) \le 0. \tag{97}$$

Note that $-z \ge 0$.

Let us take into account the fact that in study [3, 4] the authors make assumptions (87)-(88). It is clear that condition (95) will be fulfilled if the following inequality satisfies:

$$-\varepsilon_i + \varepsilon_1^M - \frac{z}{E_1} \le 0. \tag{98}$$

The latter inequality is identical to condition:

$$\sigma(\varepsilon_i) \ge \sigma(\varepsilon_1^M). \tag{99}$$

In [3, 4] the authors obtain the following expression for strain $\varepsilon_i$:

$$\varepsilon_i = \varepsilon_* + \frac{\varepsilon_3^m - \varepsilon_*}{2} + \left( \left( \frac{\varepsilon_3^m - \varepsilon_*}{2} \right)^2 + \left( \frac{2v_0}{c_e} - \varepsilon_* \right)^2 \right)^{1/2}. \tag{100}$$

Here they assume that strain $\varepsilon_e$ is given by Eqs. (87)-(88).

Substituting Eq. (100) for $\varepsilon_i$ into inequality (98), one obtains:

$$\varepsilon_* + \frac{\varepsilon_3^m - \varepsilon_*}{2} + \left( \left( \frac{\varepsilon_3^m - \varepsilon_*}{2} \right)^2 + \left( \frac{2v_0}{c_e} - \varepsilon_* \right)^2 \right)^{1/2} \geq \varepsilon_1^M - \frac{z}{E_1}. \tag{101}$$

Eq. (88) leads to:

$$-\frac{z}{E_1} = \varepsilon_3^m - \varepsilon_* \tag{102}$$

Taking into account the above equation we can rewrite inequality (101) in the form:

$$\frac{\varepsilon_3^m - \varepsilon_*}{2} + \left( \left( \frac{\varepsilon_3^m - \varepsilon_*}{2} \right)^2 + \left( \frac{2v_0}{c_e} - \varepsilon_* \right)^2 \right)^{1/2} \geq \varepsilon_1^M - \varepsilon_*. \tag{103}$$

It is clear that $\varepsilon_3^m - \varepsilon_* > 0$. Hence, inequality (103) can be rewritten as follows:

$$1 + \left( 1 + \frac{4(2v_0/c_e - \varepsilon_*)^2}{(\varepsilon_3^m - \varepsilon_*)^2} \right)^{1/2} \geq 2\frac{\varepsilon_1^M - \varepsilon_*}{\varepsilon_3^m - \varepsilon_*}. \tag{104}$$

One can see that the left-hand side of the above inequality is greater than 2 (or is equal to 2). The right-hand side of the inequality is less than 2 (or equals 2). This means that the inequality can be satisfied for all admissible values of its parameters. Hence, thermodynamical condition in the form of inequality (14) fulfills.

For the case when the stiffness of the inner phase equals to zero in [3, 4] Bažant et al. consider the stress-strain dependency with $E_3 = 0$, $\sigma(\varepsilon_3^m) > 0$. Since they make an assumption (87),

$$\sigma(\varepsilon_e) = \sigma(\varepsilon_*) = \sigma(\varepsilon_3^m) = \sigma(\varepsilon_i). \tag{105}$$

Taking into account the above equation, from Hugoniot condition (13) one can see that phase boundary speed

$$\dot{l} = 0. \tag{106}$$

Another formulation for the strain-softening effect is proposed by Bažant and Belytschko in [2]. In this paper they consider the stress-strain diagram with the decreasing part and treat the case when the new phase corresponds to the incident part of the diagram. In section 6 we have demonstrated that transitions to phase 2, corresponding to the incident part of the stress-strain curve, are energetically impossible.

# 8 Conclusion

In the paper we considered the dynamic processes in a bar with with strain-softening from the standpoint of the modern theory of phase transitions. In fact, we consider the process of the new phase nucleation as a result of two shock waves collision in a bar made of a material with non-convex strain energy potential. This means that the stress-strain diagram for the material of the bar has a negative slope. As a result of the treatment have formulated the necessary conditions for a new phase nucleation, found out the expressions for strains at the phase boundary, by taking into account additional thermodynamical condition we have derived the algebraic Eq. (45) for a phase boundary speed. Eq. (45) can be solved numerically. The results of numerical simulation shows that the solutions satisfying all necessary restrictions exist. Thus, the scenario of a new phase nucleation, proposed in this study, can be realised. By means of system for analytical simulations Maple 12 the analytical solution of Eq. (45) is derived. However, the solution is rather complex, with multiple parameters and non-informative. Therefore, have considered some important particular cases for which we have obtained the closed-form solutions.

In section 7 we discuss the solutions obtained by Bažant et al. in [2–4] for the problem in the formulation similar to the one considered in the present paper. At the same time Bažant et al. do not talk in terms of phase transitions in their studies and do not consider thermodynamical condition (14) as an additional constitutive equation which is necessary to formulate the fully defined problem. An additional condition in the form of Eq. (87), proposed in [3, 4] does not seem to be well enough justified and general. Obviously, this assumption is not valid for some particular cases, for example, for the case when $\sigma(\varepsilon_3^m) = 0$.


# Acknowledgements

The authors are very grateful to Prof. A. Metrikine who drew our attention to the studies [2, 3, 4]. This work is supported by the Russian Foundation for Basic Research (project 11-01-00385-a).


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