Motion of a Rigid Body Consisting of Two Symmetric Laminae on a Horizontal Plane

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Abstract

The problem of motion of a rigid body consisting of two equal symmetric laminae whose symmetry planes are at right angle is investigated. All equilibria of the body on the plane are found and their stability analysis is performed.

1 Introduction

Let us consider the rigid body of the following shape: it comprises of two symmetric laminae whose planes of symmetry make a right angle between each other. The laminae are connected along the common axis of symmetry. When this body moves along the fixed horizontal plane it touches the plane in two points (Fig. 1).

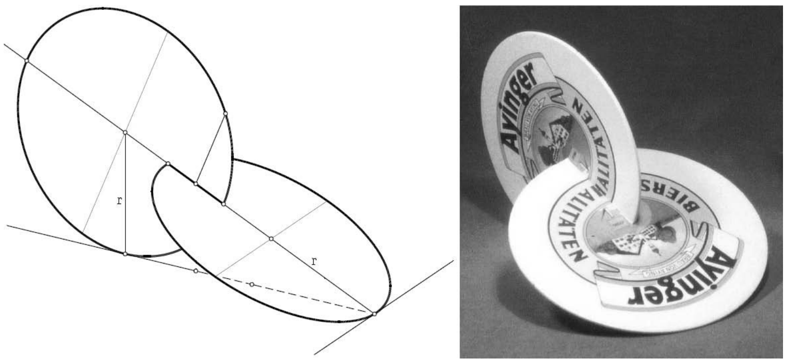

Figure 1: The Two-Circle-Roller and the Oloid.

The most known bodies of such a shape are the Two-Circle-Roller [1, 2] and the Oloid [3, 4]. The Two-Circle-Roller consists of two interlocked circular disks with the distance between their centers is $r\sqrt{2}$, where r is a common radius of the disks (Fig. 1). The Oloid is similar to Two-Circle-Roller: it consists of two interlocked circular disks but the distance between their centers equals to their radius r. For both of these bodies their motion on a fixed horizontal plane has been studied in details [1]-[4]. However it is interesting to investigate the motion of the rigid body whose form differs from Two-Circle-Roller and Oloid.

The theory proposed in [4] allows to investigate the motion of the rigid body when this body comprises of two symmetric laminae of the arbitrary shape. Using this theory the motion of the rigid body consisting of two elliptical disks has been investigated [5].

In this paper we derive the conditions of existence and stability of equilibria of the body consisting of two symmetric laminae of the arbitrary shape. The correctness of the obtained conditions is verified for the set of particular cases – for example in the case of motion of a rigid body consisting of two symmetrical elliptical disks on a horizontal plane [5].

2 Problem Formulation

Let the rigid body consisting of two equal symmetric laminae moves along the fixed horizontal plane. The planes of symmetry of the laminae make a right angle between each other. Let the distance between the centers of mass C_1 and C_2 of the laminae equals 2Δ and the center of mass G of the rigid body is situated at the middle of C_1C_2: $GC_1 = GC_2 = \Delta$. According to the theory discussed in [4, 5] let us introduce the moving coordinate frame $Gx_1x_2x_3$. The origin of this frame will be at the center of mass G of the moving body. The Gx_3 – axis is perpendicular to the plane Π_1 of the first lamina, Gx_1 – axis is perpendicular to the plane Π_2 of the second lamina and Gx_2 axis is directed along the common axis of symmetry of two symmetric laminae (Fig. 2). The unit vectors of this coordinate system will be $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

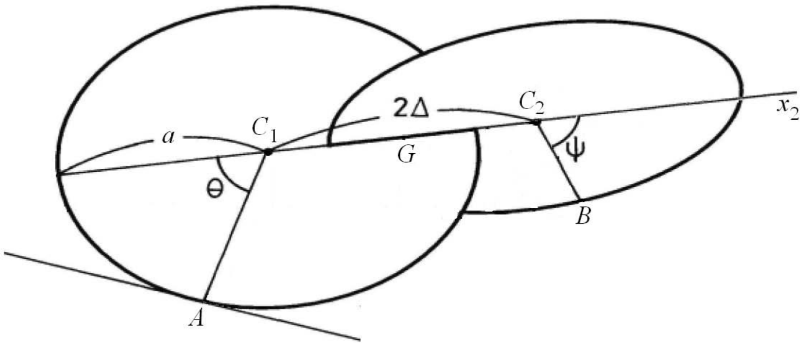

Figure 2: Rigid body consisting of two symmetric laminae.

Let A and B are two points of contact of the body with the supporting plane. We will define the position of the point A by the angle θ between the negative direction of Gx_2 axis and the direction C_1A from the center of mass C_1 of the first lamina to the point of contact A (Fig. 2). We shall assume that the shape of the lamina is completely defined if the dependence of C_1A on θ is given: $C_1A = r(\theta)$. Similarly the position of the point B is defined by the angle ψ between the positive direction of Gx_2 axis and the direction C_2B from the center C_2 of the second lamina to the point of contact B (Fig. 2). The shape of the second lamina is defined by the function $C_2B = r(\psi)$. Since the laminae have the same shape we shall assume that the distances C_1A and C_2B are defined by the same function depending on the different arguments: $C_1A = r(\theta)$, $C_2B = r(\psi)$. Then the radius - vector of the point A can be written as follows:

$$\overrightarrow{GA} = \boldsymbol{r}_1 = r(\theta)\sin\theta\boldsymbol{e}_1 - \big(\Delta + r(\theta)\cos\theta\big)\,\boldsymbol{e}_2.$$

and the radius-vector of the point B has the form:

$$\overrightarrow{GB} = \boldsymbol{r}_2 = \big(\Delta + r(\psi)\cos\psi\big)\,\boldsymbol{e}_2 - r(\psi)\sin\psi\boldsymbol{e}_3.$$

Since the considered system has only one degree of freedom the variables θ and ψ are not independent. Let us find the relation between θ and ψ. When the considered rigid body rolls on a fixed horizontal plane the three vectors $\boldsymbol{r}_2 - \boldsymbol{r}_1$, $(\boldsymbol{r}_1)'_\theta$ and $(\boldsymbol{r}_2)'_\psi$ are always in this plane. We can write this condition as follows:

$$< \boldsymbol{r}_2 - \boldsymbol{r}_1,\, (\boldsymbol{r}_1)'_\theta,\, (\boldsymbol{r}_2)'_\psi >= 0,$$

where $< \cdot, \cdot, \cdot >$ is a triple scalar product of these vectors. From this condition we can find the following symmetrical formula connecting θ and ψ:

$$\begin{aligned} &\big(r'(\psi)\sin\psi + r(\psi)\cos\psi\big)\,\Phi(\theta) + \big(r'(\theta)\sin\theta + r(\theta)\cos\theta\big)\,\Phi(\psi) = 0, \\ &\Phi(x) = r^2(x) + \Delta r(x)\cos x + \Delta r'(x)\sin x. \end{aligned} \tag{1}$$

By analyzing the formula (1) it is possible to prove the following statement.

Statement 1. *The variables θ and ψ cannot vanish simultaneously.*

Proof. Indeed if $\psi = \theta = 0$ then equation (1) takes the form:

$$r^2(0)\big(\Delta + r(0)\big) = 0.$$

Since $r(\theta) > 0$ for all admissible values of θ (i.e. for the values of θ such that the motion of the body on the plane is possible) we conclude that equation (1) cannot be valid if $\psi = \theta = 0$. □

Further we shall assume that it is possible to find the function $\psi = \psi(\theta)$ from the equation (1).

3 Potential energy of the body. Equilibria of the body

Let us derive now equation for the fixed plane in the $Gx_1x_2x_3$ coordinate system. This equation can be derived from the condition that the points A, B and the tangent vector to the first lamina at A are always in the fixed plane. Therefore the corresponding equation can be written as follows:

$$\big(r'(\theta)\cos\theta - r(\theta)\sin\theta\big)X + \big(r'(\theta)\sin\theta + r(\theta)\cos\theta\big)Y -$$

$$-\frac{\big(r'(\psi)\cos\psi - r(\psi)\sin\psi\big)\Phi(\theta)}{\Phi(\psi)}Z + \dots = 0.$$

The unit vector

$$\boldsymbol{n} = \frac{\Phi(\psi)}{\sqrt{\Big(r^2(\theta) + \big(r'(\theta)\big)^2\Big)\Phi^2(\psi) + \big(r'(\psi)\cos\psi - r(\psi)\sin\psi\big)^2\Phi^2(\theta)}} \times$$

$$\times\Bigg[\big(r'(\theta)\cos\theta - r(\theta)\sin\theta\big)\boldsymbol{e}_1 + \big(r'(\theta)\sin\theta + r(\theta)\cos\theta\big)\boldsymbol{e}_2 -$$

$$-\frac{\big(r'(\psi)\cos\psi - r(\psi)\sin\psi\big)\Phi(\theta)}{\Phi(\psi)}\boldsymbol{e}_3\Bigg]$$

is the normal vector to this plane. Having expressions for the vector $\overrightarrow{GA} = \boldsymbol{r}_1$ and the normal vector $\boldsymbol{n}$ to the fixed plane we can easily find expression for the potential energy of the body consisting of two symmetric laminae:

$$V = Mgz_G = -Mg\left(\overrightarrow{GA}\cdot\boldsymbol{n}\right).$$

Here M is the mass of the moving body; g is the acceleration due to gravity. In the explicit form the potential energy can be written as follows:

$$V = \frac{Mg\Phi(\theta)\Phi(\psi)}{\sqrt{\left(r^2(\theta) + \left(r'(\theta)\right)^2\right)\Phi^2(\psi) + \left(r'(\psi)\cos\psi - r(\psi)\sin\psi\right)^2\Phi^2(\theta)}}.$$

Critical points of the potential energy correspond to the equilibria of the body. They are determined from the equation

$$\begin{aligned}
&\left(r(\theta)r'(\theta) + \Delta r'(\theta)\,cos\theta - \Delta r(\theta)\sin\theta\right)\Phi(\psi)r(\psi)\sin\psi +\\
&+\frac{\Phi(\theta)r(\theta)r(\psi)\sin\theta\sin\psi}{2}\left(r'(\psi)\sin\psi + r(\psi)\cos\psi\right) -\\
&-\left(r(\psi)r'(\psi) + \Delta r'(\psi)\,cos\psi - \Delta r(\psi)\sin\psi\right)\Phi(\theta)r(\theta)\sin\theta -\\
&-\frac{\Phi(\psi)r(\theta)r(\psi)\sin\theta\sin\psi}{2}\left(r'(\theta)\sin\theta + r(\theta)\cos\theta\right) = 0.
\end{aligned} \tag{2}$$

By analyzing the equation (2) it is possible to prove the following statements concerning the equilibria of the body on the plane.

Statement 2. *The rigid body has the equilibrium*

$$\psi = \theta$$

when θ is satisfied the condition

$$\Phi(\theta)\left(r'(\theta)\sin\theta + r(\theta)\cos\theta\right) = 0. \tag{3}$$

Proof. It is easy to see that condition (2) is valid when $\theta = \psi$. Therefore the equilibrium $\psi = \theta$ exists for any value of θ which satisfied the condition (1). If we substitute $\psi = \theta$ to the condition (1) it is reduced to the condition (3). In other words the equilibrium $\psi = \theta$ exists only for any value of θ which satisfied the condition (3). □

Statement 3. *The equilibrium $\theta = 0$ exists if and only if $\theta = 0$ is an extreme point of the function $r(\theta)$.*

Proof. Indeed the substitution of the value $\theta = 0$ to the equation (2) reduces this equation to the form:

$$r'(\theta)\,|_{\theta=0}\left(\Delta + r(0)\right)\left(\Phi(\psi)r(\psi)\sin\psi\right)|_{\theta=0} = 0.$$

According to the Statement 1, the expression $\Phi(\psi)r(\psi)\sin\psi$ cannot vanish when $\theta = 0$. Therefore the rigid body has an equilibrium at $\theta = 0$ if $r'(\theta)\,|_{\theta=0} = 0$. □

Further we will assume that the function $r(\theta)$ is an even function and it has a minimum value at $\theta = 0$. Taking into account this assumptions we can prove the following statement.

Statement 4. *If $r(\theta)$ is an even function and θ is satisfied the condition* (3) *then the system has the equilibrium $\psi = -\theta$ in addition to the equilibrium $\psi = \theta$.*

Proof. Indeed if $r(\theta)$ is an even function then $r'(\theta)$ is an odd function and $r'(\theta)\sin\theta$ is again an even function. Thus all the functions in the equation (1) are an even functions and equation (1) does not change its form if we change θ by $-\theta$. Similarly equation (2) does not change its form if we change θ by $-\theta$. This proves our statement. □

4 Stability of equilibria

Stability of the found equilibria of the rigid body on the plane is determined by the sign of the second derivative of the potential energy V calculated at the corresponding equilibria. By the direct calculation of the second derivative of V at the equilibrium $\psi = \theta$ we obtain the following result.

Statement 5. *The equilibrium $\psi = \theta$ is stable when the condition*

$$2r(\theta)\sin\theta - \left(r^2(\theta)\sin^2\theta + 2\left(\Delta + r(\theta)\cos\theta\right)^2\right)K > 0$$

is valid. Here K is the curvature of the boundary of the lamina at the corresponding equilibrium.

Thus we can conclude that the equilibrium $\psi = \theta$ is stable for the sufficiently small values of Δ, i.e. when the centers of mass of the laminae are close to each other.

Let us find now the stability conditions of equilibrium $\theta = 0$. Note that $\psi = \psi_0$ when $\theta = 0$ and ψ_0 is determined from the equation

$$r^2(\psi_0) + \left(r(0) + 2\Delta\right)\left(r'(\psi_0)\sin\psi_0 + r(\psi_0)\cos\psi_0\right) = 0.$$

We can find $r'(\psi_0)$ from this equation and use it in calculating of the second derivative of the potential energy at $\theta = 0$. As a result it is possible to prove the following statement.

Statement 6. *The equilibrium $\theta = 0$ is stable under the condition:*

$$\begin{aligned} &2\Delta^2 + \left(r(0) + 3r(\psi_0)\cos\psi_0 - Kr^2(\psi_0)\sin^2\psi_0\right)\Delta + \\ &+ r(\psi_0)\left(r(0)\cos\psi_0 + r(\psi_0) - Kr(0)\,r(\psi_0)\sin^2\psi_0\right) > 0, \end{aligned} \tag{4}$$

where

$$K = \frac{r(0) - r''(0)}{r^2(0)}$$

is the curvature of the boundary of the lamina at $\theta = 0$.

Condition (4) has a form of the polynomial of the second degree with respect to Δ. Since Δ^2 grows faster than Δ we can conclude that the equilibrium $\theta = 0$ will be stable for the sufficiently large values of Δ.

The obtained results on the stability of equilibria of the considered rigid body on the plane have been successfully verified in particular cases of the body consisting of two circular disks [4] and two elliptical disks [5].

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