The threshold extent of softening and the incipiency of fracture in the Tresca-type materials

Tatyana B. Lavrova
lavr@samsu.ru

Abstract

The initial diffuse microfracturing of a solid within the limits when it is still can be considered as a continuum, is expressed phenomenologically by a law of plasticity with softening, the latter in principle enabling to describe initial stages of fracturing within the framework and using the methods of solid mechanics. In the work there are studied the bounds of stable processes of diffuse microfracturing, i.e. of the deformation processes under softening for the Tresca-type elastic-plastic materials. It is known [1] that the bounds of stable deformation are established by the Hadamard inequality for the moduli tensor of plastic response: within the range of validity of the inequality deformation is stable (under certain additional conditions), whereas violation of the inequality is surely followed by the onset of instability in the form of localized instability. The latter is treated as the incipiency of macroscopic fracture in a solid. Hardening of the material means that all its true moduli are positive, and softening means that at list one is negative. Thus, hardening is equivalent to positive definiteness of Drucker's quadratic form, and softening means that it can take negative values (just Drucker's definition of hardening/softening [2]). The analysis presented here is based upon the assumption that initial stresses are negligibly small as compared to the elastic moduli. In that case, the Hadamard inequality reduces to positive semi-definiteness of Drucker's quadratic form on the set of symmetrized dyads, the latter being the part of the whole set of symmetric second-order tensors. Hence, the Hadamard inequality can be valid even in the case when Drucker's quadratic form takes negative values, but not on the symmetrized dyads. In other words, softening within certain bounds is compatible with the Hadamard inequality. In determining such bounds of admissible softening for the Tresca-type elastic-plastic materials, it is of principle significance to take into account that the yield surface possesses edges. The normals to the faces of yield surface (in the strain space) are symmetrized dyads, and as for the faces adjacent to an edge, corresponding normals are the co-axial symmetrized dyads. On the edges, there are four regimes of constitutive response: elastic unloading, two regimes of partial plastic loading and that of complete plastic loading. It is shown in the work that both on the faces of yield surface and on the edges for regimes of partial loading the onset of softening coincides with violation of the Hadamard inequality, that is just the condition for origin of the localized instability, which is treated as the incipiency of macroscopic fracture in a solid. It is proved also that in the regime of complete plastic loading there exists certain finite achievable extent of softening, that being threshold softening for origin of the localized instability and for incipiency of fracture. It is developed an algorithm of determination of the both the threshold value of softening and the mode of incremental strains in the zones of localization together with spatial orientation of the zones themselves. The latter are just future displacement discontinuities in embryo.

1 Introduction

From the viewpoint of phenomenological approach to description of initial stages of fracture of a solid, softening is an expression of diffuse microfracturing, whereas the localized instability is the initial stage of macroscopic fracturing, the zones of localized incremental deformation being treated as future macroscopic displacement discontinuities in embryo. With this approach, investigation of the initial stage of macroscopic fracture reduces to determination of the moment of onset and the character of localized instability in a solid. As a rule, the studies of that type were carried out for the elastic-plastic materials with smooth yield surfaces (in the strain space). However, some essential properties of elastic-plastic materials can be expressed only by means of the laws of elastic-plastic behavior with nonsmooth yield surfaces. In particular, the laws of such a type are those with the Tresca-type yield surfaces, and this very type of elastic-plastic laws is studied here. It is known that a process of uniform deformation of elastic-plastic solid under some special conditions is stable up to violation of the Hadamard inequality for the moduli tensor of a material, whereas violation of the inequality is followed immediately by instability, that being the localized one. Note that for elastic material at any stage there is the single moduli tensor, for elastic-plastic material with smooth yield surface (in plastic state) there are two such moduli tensors (for elastic unloading and for active plastic deformation), and for the Tresca-type materials at the state, corresponding to the yield surface edge, there are four such tensors (for unloading, for two regimes of partial plastic deformation and for complete plastic deformation). In the cases when there are more than one moduli tensor, by violation of the Hadamard inequality is meant its violation for that moduli tensor, for which it takes place earlier than for the others. Due to the properties of elastic-plastic constitutive relations with the smooth yield surface, it is the moduli tensor for plastic response of the material, and for the Tresca-type materials on the edge of yield surface it is the moduli tensor of complete plasticity response.

2 The Tresca-type materials

For the case of non-pure plasticity, when at first it takes place hardening and then softening, the yield surface is varied during the process of plastic deformation. The variation depends essentially on the deformation path and is specified not only by the current values of strain and plastic strain tensors, but additionally by a scalar parameter hereafter called the history parameter q. The natural requirement of continuity of plastic response on the edge reduces to the requirement of mutually related motion of yield surfaces corresponding to the faces whose intersection forms the edge. This results in elimination of arbitrariness in possible history parameters q. The yield surface for the class of materials in hand is of the form [3]:

$$\max_{m\neq n}\left|\Big(\mathbf{L}^H : (\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p) - q\boldsymbol{\varepsilon}^p\Big) : (\mathbf{e}_n\otimes\mathbf{e}_n - \mathbf{e}_m\otimes\mathbf{e}_m)\right| - k(q) = 0,\ (m,n) = 1,2,3. \quad (1)$$

Initially it coincides with the Tresca prism, transited to the strain space, subsequently it changes its size (that is specified by the function $k(q)$) and shape so that the faces can become non-flat (remaining convex) and the edges remain straight. We use the following notation: $\boldsymbol{\varepsilon}$ is the small strain tensor, $\boldsymbol{\varepsilon}^p$ is the plastic strain tensor, $\mathbf{1}^{dev}$ is the fourth-order tensor that maps every second-order tensor into its symmetrized deviator, $\mathbf{E}$ is the unit second-order tensor, $\mathbf{L}^H$ is the elastic moduli tensor identical to that of classical Hook's law:

$$\mathbf{L}^H = 2G\mathbf{1}^{dev} + K\mathbf{E}\otimes\mathbf{E},$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the eigen-vectors of the second-order tensor $\mathbf{L}^H : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - q\boldsymbol{\varepsilon}^p$, G is the shear modulus, and K is the bulk modulus. The properties of particular material are specified by the moduli G and K and the function $k(q)$ (obtained basing on data of the uniaxial extension experiment). The moduli tensor for plastic response on the faces of yield surface (1) is given by the equality

$$\mathbf{L}^p = 2G\mathbf{1}^{dev} + K\mathbf{E}\otimes\mathbf{E} - \theta\mathbf{N}\otimes\mathbf{N}, \tag{2}$$

where $\mathbf{N}$ is normal to the face, $\theta\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, q\right)$ is positive scalar function of the material state, the presence of the properties of hardening or softening being governed by its value. For the yield surface in hand, its normal $\mathbf{N}$ is simultaneously a symmetrized dyad and a deviator.

The moduli tensors of plastic response in the states corresponding to the edge of yield surface (1) are determined by the fact that the edge is intersection of two faces. Their tangent hyperplanes at the edge point form four dihedral angles, those results in existence of four regimes of incremental material response. The moduli tensors for partial plasticity coincide with those for plastic response on the adjacent face. The moduli tensor for regime of complete plasticity is given by the equality

$$\mathbf{L}^p = 2G\mathbf{1}^{dev} + K\mathbf{E}\otimes\mathbf{E} - \tilde{\theta}\left(\mu'\mathbf{N}_1\otimes\mathbf{N}_1 + \mu''\mathbf{N}_2\otimes\mathbf{N}_2\right), \quad \mu' + \mu'' = 1, \tag{3}$$

where $\tilde{\theta}$, μ', μ'' are positive scalar functions of the material state: $\tilde{\theta}$ specifies the extent of hardening or softening, and μ', μ'' depend on gained plastic strain. The tensors $\mathbf{N}_1$ and $\mathbf{N}_2$ are normals to the adjacent faces (deviatoric symmetrized dyads). In addition, they are co-axial and non-orthogonal to each other.

3 The threshold extent of softening

The state of material is called either that of hardening, if Drucker's quadratic form $d\boldsymbol{\varepsilon} : \mathbf{L}^p : d\boldsymbol{\varepsilon}$ (defined on the space of symmetric second-order tensor) is positive definite, or that of softening, if for at least one $d\boldsymbol{\varepsilon}^*$ it takes negative value. The fourth-order tensor $\mathbf{L}^p$ is a symmetric linear operator on the six-dimensional space of symmetric second-order tensors. Due to spectral theorem, it possesses the orthonormal basis of six second-order eigen-tensors and corresponding set of six eigen-values (called "true moduli" according to terminology of J. Rychlewski [4]). The positive definiteness of the form is equivalent to positivity of all true moduli, whereas if only if there is at least one negative true modulus, the form takes negative values. Hence, the softening originates when the least true modulus first becomes negative. From (2) it is straightforward that both on a face and on an edge in regime of partial plasticity the tensor $\mathbf{N}$ is the eigen-tensor, the corresponding true modulus $\alpha = 2G - \theta$ being the least one. Thus, in both cases softening takes place for negative values of α, which means that $\theta \geq 2G$. However, in the case of complete plasticity neither $\mathbf{N}_1$ nor $\mathbf{N}_2$ are the eigen-tensors of the $\mathbf{L}^p$. The eigen-tensor that corresponds to the least true modulus is a linear combination of the tensors $\mathbf{N}_1$ and $\mathbf{N}_2$, which enables to seek it on the corresponding invariant two-dimensional linear space making use of a standard method (which is not much complicated due to low dimension of the space). There are two eigen-tensors:

$$\mathbf{M}_1 = x_1\mathbf{N}_1 + y_1\mathbf{N}_2 \text{ and } \mathbf{M}_2 = x_2\mathbf{N}_1 + y_2\mathbf{N}_2, \text{ where } x_1 = \frac{2(1+\nu)}{\sqrt{f_1}}, \nu = \mu' - \mu'', \tag{4}$$

$$y_1 = \frac{\sqrt{1+3\nu^2}/2-\nu}{\sqrt{f_1}},\ f_1 = 1.75\nu^2 + \nu + 1.25 + 0.5(1-\nu)\sqrt{1+3\nu^2},\ x_2 = \frac{2(1+\nu)}{\sqrt{f_2}},$$
$$y_2 = \frac{\sqrt{1+3\nu^2}/2-\nu}{\sqrt{f_2}},\ f_2 = 1.75\nu^2 + \nu + 1.25 - 0.5(1-\nu)\sqrt{1+3\nu^2}.$$

Depending on the sign of ν, the least true modulus may correspond either to $\mathbf{M}_1$ or to $\mathbf{M}_2$: if $\nu > 0$ then $\alpha_1 = \mathbf{M}_1 : \mathbf{L}^p : \mathbf{M}_1 = 2G - (\tilde{\theta}/8)\varphi_1(\nu)$ is the least true modulus, and if $\nu < 0$ then $\alpha_2 = \mathbf{M}_2 : \mathbf{L}^p : \mathbf{M}_2 = 2G - (\tilde{\theta}/8)\varphi_2(\nu)$ is the least one ($\varphi_{1,2} = 5 + 11\nu + 17\nu^2 - 3\nu^3 \pm 4\nu(\nu-1)\sqrt{1+3\nu^2}$). Knowing the least true modulus (whose negativity characterizes the states of softening) and corresponding second-order eigen-tensor, we proceed to investigating the realizability of such states and determining the threshold extent of softening. The investigation is based upon two fundamental theorems of the mechanics of solids: the Hadamard stability theorem and the Van Hove theorem (see e.g. [1]), the latter in a sense inverting assertion of the former. The Hadamard theorem states that violation of the Hadamard inequality for material anywhere in a body results in its instability under any boundary conditions. Hence, the material states for which the Hadamard inequality is violated are not realizable. The Van Hove theorem (in couple with some its modification) in fact asserts that homogeneous states of a body under sufficiently stiff boundary constraint are stable up to violation of the Hadamard inequality. Thus, investigation of bounds of the realizable in principle states of softening reduces to investigation of the bounds of compatibility of softening with the Hadamard inequality. Note that in analysis, we restrict ourselves to the case of small current stresses as compared to elastic moduli, and hence, the Hadamard inequality reduces to non-negativity of the quadratic form, corresponding to the $\mathbf{L}^p$ tensor, on dyads. Due to the properties of the tensor $\mathbf{L}^p$, it is equivalent to non-negativity of the above-mentioned form on the set of symmetrized dyads. Note that the set of symmetrized dyads is only a part of the whole set of symmetric second-order tensors, whence it follows that the Hadamard inequality can be valid even when on the whole set of symmetric tensors there are some giving negative values to the form $d\boldsymbol{\varepsilon} : \mathbf{L}^p : d\boldsymbol{\varepsilon}$. In other words, softening is in principle compatible with the Hadamard inequality, but not always and within certain bounds.

On a face as well as for regimes of partial plasticity on an edge, the second-order eigen-tensor corresponding to the least true modulus (the normal to the face) is a symmetrized dyad; hence, the origin of softening coincides with violation of the Hadamard inequality, i.e. with the onset of localized instability. The normal mentioned is expressed by equality $\mathbf{N} = \mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_j \otimes \mathbf{e}_j$. We introduce vectors $\mathbf{n} = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}$ and $\mathbf{g} = (\mathbf{e}_i + \mathbf{e}_j)/\sqrt{2}$; then $\mathbf{N} = \mathbf{n} \otimes \mathbf{g} + \mathbf{g} \otimes \mathbf{n}$. In that case, arising instability is characterized by localization of incremental strain in a shear layer, its plane having the normal $\mathbf{n}$ and shears direction $\mathbf{g}$ or vice versa. On an edge in the regime of complete plasticity the second-order eigen-tensor of $\mathbf{L}^p$, corresponding to the least true modulus, is not the symmetrized dyad, and so when this modulus vanishes, the Hadamard inequality is still valid strictly, whereas it first violates at a finite negative value of the modulus. For the tensor $\mathbf{L}^p$ specified by (3), non-negativity of Drucker's quadratic form is equivalent to inequality

$$\tilde{\theta} \le 2G\left((\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}) + \beta\,(\mathbf{E} : \boldsymbol{\varepsilon})^2\right) / \left(\mu'\,(\mathbf{N}_1 : \boldsymbol{\varepsilon})^2 + \mu''\,(\mathbf{N}_2 : \boldsymbol{\varepsilon})^2\right),\ \beta = K/2G \tag{5}$$

for every $\boldsymbol{\varepsilon}$ from the corresponding set (here it is the set of symmetrized dyads). The threshold (for the onset of localized instability) value $\tilde{\theta}^*$ is just the minimum of the right hand side of (5) on the set mentioned, that is specified by the following conditions:

$$det\boldsymbol{\varepsilon} = 0, \quad (\mathbf{E} : \boldsymbol{\varepsilon})^2 - \mathbf{E} : \boldsymbol{\varepsilon}^2 \le 0. \tag{6}$$

Setting additionally the denominator of the right hand side of (5) to be equal to unity:

$$\mu'\,(\mathbf{N}_1 : \boldsymbol{\varepsilon})^2 + \mu''\,(\mathbf{N}_2 : \boldsymbol{\varepsilon})^2 = 1, \tag{7}$$

we reduce search of minimum for the right hand side mentioned to search of minimum for its numerator under the specified conditions. Thus, we find both the threshold extent of softening and corresponding value of the strain tensor $\boldsymbol{\varepsilon}$ (the mode of incremental strain in the zones of localized deformation), the latter giving two possible spatial orientations of the zones themselves. For example, for the edge of compression of the yield surface (1) the normals to adjacent faces are given by $\mathbf{N}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_3 \otimes \mathbf{e}_3$, $\mathbf{N}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3$ and for the threshold value $\tilde{\theta}^*$ it is obtained the equality

$$\tilde{\theta}^* = 2G\left(\frac{1+2\beta}{\mu'} + \frac{1+\beta}{2\mu''} - \sqrt{\left(\frac{1+\beta}{2\mu''} - \frac{1+2\beta}{\mu'}\right)^2 + \frac{(1+2\beta)^2}{\mu'\mu''}}\right)$$

(it is implied here that $\mu' > \mu''$, and otherwise μ' and μ'' should be transposed). In that case, the instability mode is characterized by the localization of incremental strain in a plane layer. The vectors $\mathbf{n}$ and $\mathbf{g}$ (or vice versa, $\mathbf{g}$ and $\mathbf{n}$) are the normals to boundary planes of a layer and their normalized relative shift, respectively. The vectors are given by

$$\begin{aligned}
\mathbf{n} &= \sqrt{1-\gamma}\left(\mathbf{e}_2 + \sqrt{1/\gamma - 1}\mathbf{e}_3\right), \quad \mathbf{g} = \sqrt{1-\gamma}\left(\mathbf{e}_2 - \sqrt{1/\gamma - 1}\mathbf{e}_3\right), \\
\gamma &= \sqrt{\left(\mu''\left(1-\cos\alpha\right)\right) / \left(\mu'\left(1+\cos\alpha\right)\right)}, \\
\cos\alpha &= \left(1+2\beta\right) / \left(\left(1+\beta\right)^2 \mu' / \left(4\mu''\right) + \beta\left(1+2\beta\right) + \left(1+2\beta\right)^2 \mu''/\mu'\right)^{1/2}.
\end{aligned}$$

As for the other edges of a yield surface, the quantities $\tilde{\theta}^*$, $\mathbf{n}$ and $\mathbf{g}$ are calculated for them in a quite similar way.

References

[1] E.I. Ryzhak: A case of indispensable localized instability in elastic-plastic solids. Int. J. Solids Structures. Vol.36, No.30, 4669–4691 (1999).

[2] D.C. Drucker: A definition of stable inelastic material. J. Appl. Mech. ASME. Vol. 26, 101–106 (1959).

[3] G.I. Bykovtsev, T.B. Lavrova: Properties of singular yield surfaces in strain space. In: Applied Problems of the Mechanics of Solids. 3–20, DVO RAN, Vladivostok (1991) (in Russian).

[4] J. Rychlewski: On Hook's law. Pricl. Mat. Mekh. Vol.48, No.3, 420–435 (1984) (in Russian).

Tatyana B. Lavrova, Samara State University, acad. Pavlova, 1, 443011, Samara, Russia