On the problem of forced plane vibrations of transversally inhomogeneous elastic layer

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Abstract

A problem of forced plane vibration of the transversally inhomogeneous elastic layer is considered. A calculation scheme for the wave field evaluation is presented. It's based on the Fourier transform and the initial problem is reduced to the boundary value problem. This problem can be solved numerically using the shooting method. Then the wave field can be evaluated using the residual theory, or it can be obtained using numerical integration methods.

An inverse problem of mechanical parameters reconstruction using surface wave field data is also considered. The inverse problem is reduced to the iterative sequence of integral equations. Results of both direct and inverse problem solution are presented.

1 Problem statement

Consider the elastic layer forced vibrations. The layer occupies the domain $\{(x_1, x_2) : -\infty < x_1 < \infty, 0 \le x_2 \le h\}$, where h is the layer thickness.

Vibrations satisfy the following equations system

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} + \rho\omega^2 u_1 = 0, \\ \sigma_{12,1} + \sigma_{22,2} + \rho\omega^2 u_2 = 0, \end{cases} \tag{1}$$

where ρ is the layer density, ω is the frequency of vibrations, σ_{ij} are the stress tensor components, u_i are the displacement vector components. The stress tensor is connected with the displacement vector by the governing equations

$$\begin{cases} \sigma_{11} = \lambda\left(u_{1,1} + u_{2,2}\right) + 2\mu u_{1,1}, \\ \sigma_{12} = \mu\left(u_{1,2} + u_{2,1}\right), \\ \sigma_{22} = \lambda\left(u_{1,1} + u_{2,2}\right) + 2\mu u_{2,2}, \end{cases} \tag{2}$$

where λ, μ are the Lamé parameters. Mechanical parameters depend on the transversal coordinate:

$$\lambda = \lambda(x_2),\ \mu = \mu(x_2),\ \rho = \rho(x_2).$$

The bottom boundary of the layer is restrained, and the normal load is applied on the top surface:

$$u_1(x_1, 0) = u_2(x_1, 0) = 0 \tag{3}$$

$$\sigma_{12}|_{x_2=h} = 0,\ \sigma_{22}|_{x_2=h} = p(x_1), \tag{4}$$

The statement of the problem is completed with the radiation conditions formulated on the basis of the limiting absorption principle [1].

Now we introduce dimentionless values using formulae:

$$x_i = h\bar{x}_i,\ u_i = h\bar{u}_i,\ \sigma_{ij} = \mu(0)\bar{\sigma}_{ij},$$
$$\mu = \mu(0)\bar{\mu},\ \lambda = \mu(0)\bar{\lambda},\ \kappa_2^2 = \frac{\rho\omega^2 h^2}{\mu(0)}, p(x_1) = \mu(0)\bar{p}(x_1). \tag{5}$$

Overlined values in (5) are dimentionless. The overline will be skipped hereinafter.

2 The direct problem solving

To solve the problem (1)-(4) the Fourier transform is used

$$\tilde{u}(\alpha, x_2) = \int_{-\infty}^{\infty} u(x_1, x_2) e^{i\alpha x_1} dx_1$$

Eliminating $\tilde{\sigma}_{11}$ from the transformed equations (1)-(2), we reduce (1)-(2) to the boundary value problem for the canonical system of ordinary differential equations with respect to $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{\sigma}_{12}$, $\tilde{\sigma}_{22}$

$$\begin{cases} \tilde{u}_1' = i\alpha\tilde{u}_2 + \dfrac{\tilde{\sigma}_{12}}{\mu} \\ \tilde{u}_2' = \dfrac{i\alpha\lambda\tilde{u}_1 + \tilde{\sigma}_{22}}{\lambda + 2\mu} \\ \tilde{\sigma}_{12}' = \left[4\alpha^2 \dfrac{\mu(\lambda+\mu)}{\lambda + 2\mu} - \rho\kappa_2^2\right]\tilde{u}_1 + \dfrac{i\alpha\lambda\tilde{\sigma}_{22}}{\lambda + 2\mu} \\ \tilde{\sigma}_{22}' = -\rho\kappa_2^2\tilde{u}_2 + i\alpha\tilde{\sigma}_{12}, \end{cases} \tag{6}$$

The boundary value problem (6) with boundary conditions (3)-(4) can be easily solved using the shooting method. We define:

$$U_1 = \tilde{u}_1,\ U_2 = \tilde{u}_2,\ U_3 = \tilde{\sigma}_{12},\ U_4 = \tilde{\sigma}_{22},$$

and introduce functions $V_i^j(x_2)$, satisfying the equations system (6) with initial conditions

$$V_i^j(0) = \delta_{ij},$$

where δ_{ij} is the Cronecker delta.

The solution of (6) is

$$U_i^0 = -\tilde{p}(\alpha)\frac{V_3^4(1)V_i^3(x_2) - V_3^3(1)V_i^4(x_2)}{\Delta(1)}, \tag{7}$$

$$\Delta(1) = V_3^3(1)V_4^4(1) - V_3^4(1)V_4^3(1).$$

The displacement field can be calculated using the inverse Fourier transform:

$$u_i(x_1, x_2) = -\frac{1}{2\pi}\int_\sigma \tilde{p}(\alpha)\frac{V_3^4(1)V_i^3(x_2) - V_3^3(1)V_i^4(x_2)}{\Delta(1)} e^{-i\alpha x_1} d\alpha,\ i = 1, 2. \tag{8}$$

3 The inverse problem

Now we formulate the inverse problem: determine the mechanical parameters $\lambda(x_2)$, $\mu(x_2)$, $\rho(x_2)$ distribution law from the displacement field data within a certain part of the upper boundary.

Let $\varepsilon > 0$ be a small formal parameter. Expand the displacement function and functions, describing the density and mechanical parameters, into power series of ε.

$$\begin{aligned} \lambda(x_2) &= \lambda_0(x_2) + \varepsilon\lambda_1(x_2) + ..., \\ \mu(x_2) &= \mu_0(x_2) + \varepsilon\mu_1(x_2) + ..., \\ \rho(x_2) &= \rho_0(x_2) + \varepsilon\rho_1(x_2) + ..., \\ U_i(\alpha, x_2) &= U_i^0(\alpha, x_2) + \varepsilon U_i^1(\alpha, x_2) + ... \end{aligned} \tag{9}$$

Substituting expansions (9) into equations (6) and boundary conditions (3)-(4), and collecting terms of the same order in ε, yields a boundary problem for U_i^0:

$$\begin{cases} \left[U_1^0\right]' = i\alpha U_2^0 + \dfrac{U_3^0}{\mu_0}, \\ \left[U_2^0\right]' = \dfrac{i\alpha\lambda_0 U_1^0 + U_4^0}{\lambda_0 + 2\mu_0}, \\ \left[U_3^0\right]' = \left[4\alpha^2 \dfrac{\mu_0(\lambda_0+\mu_0)}{\lambda_0+2\mu_0} - \kappa_2^2\right] U_1^0 + \dfrac{i\alpha\lambda_0 U_4^0}{\lambda_0+2\mu_0}, \\ \left[U_4^0\right]' = -\kappa_2^2 U_2^0 + i\alpha U_3^0, \\ U_1^0(\alpha,0) = U_2^0(\alpha,0) = U_3^0(\alpha,1) = 0,\ U_4^0(\alpha,1) = \tilde{p}(\alpha). \end{cases} \tag{10}$$

For U_i^1 we get the following boundary value problem

$$\begin{cases} \left[U_1^1\right]' = i\alpha U_2^1 + \dfrac{U_3^1}{\mu_0} + F_1(x_2) \\ \left[U_2^1\right]' = \dfrac{i\alpha\lambda_0 U_1^1 + U_4^1}{\lambda_0 + 2\mu_0} + F_2(x_2), \\ \left[U_3^1\right]' = \left[4\alpha^2 \dfrac{\mu_0(\lambda_0+\mu_0)}{\lambda_0+2\mu_0} - \kappa_2^2\right] U_1^1 + \dfrac{i\alpha\lambda_0 U_4^1}{\lambda_0+2\mu_0} + F_3(x_2), \\ \left[U_4^1\right]' = -\kappa_2^2 U_2^1 + i\alpha U_3^1 + F_4(x_2), \\ U_1^1(\alpha,0) = U_2^1(\alpha,0) = U_3^1(\alpha,1) = U_4^1(\alpha,1) = 0, \end{cases} \tag{11}$$

where

$$F_1(x_2) = -\frac{\mu_1}{\mu_0^2} U_3^0,$$

$$F_2(x_2) = \frac{2i\alpha}{\lambda_0+2\mu_0}\left[\frac{\mu_0(\lambda_1+2\mu_1)}{\lambda_0+2\mu_0} - \mu_1\right] U_1^0 - \frac{\lambda_1+2\mu_1}{(\lambda_0+2\mu_0)^2} U_4^0,$$

$$F_3(x_2) = \left\{ \frac{4\alpha^2}{\lambda_0+2\mu_0}\left[\lambda_0\mu_1 + \frac{\mu_0^2(\lambda_1+2\mu_1)}{\lambda_0+2\mu_0}\right] - \rho_1\kappa_2^2 \right\} U_1^0 + \frac{2i\alpha}{\lambda_0+2\mu_0}\left[\frac{\mu_0(\lambda_1+2\mu_1)}{\lambda_0+2\mu_0} - \mu_1\right] U_4^0,$$

$$F_4(x_2) = -\rho_1\kappa_2^2 U_2^0.$$

Solution of the problem (10) has the form (7).

Now we consider the problem (11) for the first approximation. According to the variation of parameters method we construct its solution in the following form:

$$U_i^1 = \sum_{k=1}^{4} C_k(x_2) V_i^k(x_2), \tag{12}$$

Substituting (12) in the boundary value problem (11), we obtain the linear equations system with respect to C_i':

$$\sum_{j=1}^{4} C_j'(x_2) V_i^j(x_2) = F_i(x_2),\ i = 1...4 \tag{13}$$

System (13) solution has the form:

$$C_i' = \sum_{j=1}^{4} F_j(x_2) A_{ji}(x_2),$$

где A_{ij} — cofactors of the system (13) matrix.

The general solution of the problem (11) now turns to

$$\begin{aligned} U_i^1 &= \sum_{k=1}^{4} V_i^k(x_2) \sum_{j=1}^{4} \int_0^{x_2} F_j(\xi) A_{jk}(\xi) d\xi + C V_i^3(x_2) + D V_i^4(x_2) = \\ &= \sum_{j=1}^{4} \int_0^{x_2} F_j(\xi) \Delta_{ij}(x_2, \xi) d\xi + C V_i^3(x_2) + D V_i^4(x_2), \end{aligned} \tag{14}$$

где

$$\Delta_{ij}(x_2, \xi) = \sum_{k=1}^{4} V_i^k(x_2) A_{jk}(\xi)$$

Now we substitute (14) to the boundary conditions of (11) to define C and D.

Then, if $x_2 = 1$, expressions for U_1 and U_2 can be reduced to:

$$\begin{aligned} U_i^1 &= \sum_{j=1}^{4} \int_0^{1} F_j(\xi) \Delta_{ij}(x_2, \xi) d\xi - \\ &- \frac{V_i^3(x_2)}{\Delta(1)} \sum_{j=1}^{4} \int_0^{1} F_j(\xi) \left[\Delta_{3j}(1, \xi) V_4^4(1) - \Delta_{4j}(1, \xi) V_3^4(1) \right] d\xi - \\ &- \frac{V_i^4(x_2)}{\Delta(1)} \sum_{j=1}^{4} \int_0^{1} F_j(\xi) \left[\Delta_{4j}(1, \xi) V_3^3(1) - \Delta_{3j}(1, \xi) V_4^3(1) \right] d\xi, \end{aligned} \tag{15}$$

Using the expression for Δ_{ij}, we derive:

$$U_i^1 = \frac{1}{\Delta(1)} \sum_{j=1}^{4} \int_0^{1} p(\alpha) F_j(\xi) \sum_{k=1}^{n} A_{jk}(\xi) \begin{vmatrix} V_i^k(1) & V_i^3(1) & V_i^4(1) \\ V_3^k(1) & V_3^3(1) & V_3^4(1) \\ V_4^k(1) & V_4^3(1) & V_4^4(1) \end{vmatrix} d\xi \tag{16}$$

Now we expand sums in the expression (16) to obtain

$$U_1^1 = \int_0^1 \tilde{p}(\alpha)\left[F_1(\xi)G_{31}(\alpha,\xi) - F_2(\xi)G_{41}(\alpha,\xi) - \right. \\ \left. -F_3(\xi)G_{11}(\alpha,\xi) + F_4(\xi)G_{21}(\alpha,\xi)\right] d\xi, \tag{17}$$

$$U_2^1 = -\int_0^1 \tilde{p}(\alpha)\left[F_1(\xi)G_{32}(\alpha,\xi) - F_2(\xi)G_{41}(\alpha,\xi) - \right. \\ \left. -F_3(\xi)G_{12}(\alpha,\xi) + F_4(\xi)G_{22}(\alpha,\xi)\right] d\xi \tag{18}$$

where

$$G_{i1} = \frac{V_4^4(1)V_i^3(x_2) - V_3^4(1)V_i^4(x_2)}{\Delta(1)}, \tag{19}$$

$$G_{i2} = -\frac{V_3^4(1)V_i^3(x_2) - V_3^3(1)V_i^4(x_2)}{\Delta(1)}, \tag{20}$$

Assume $p(x_1) = \delta(x_1)$. Substituting expressions for F_i in (17)-(18) and inverting the Fourier transform we obtain:

$$u_1^1(x_1,1) = -\int_0^1 \Big\{\mu_1(\xi)K_{11}(x_1,\xi) + \left[\lambda_1(\xi) + 2\mu_1(\xi)\right] K_{21}(x_1,\xi) - \\ -\kappa_2^2\rho_1(\xi)K_{31}(x_1,\xi)\Big\} d\xi, \tag{21}$$

$$K_{11}(x_1,\xi) = \frac{1}{2\pi}\int_\sigma \left\{ \frac{G_{32}(\alpha,\xi)G_{31}(\alpha,\xi)}{\mu_0^2(\xi)} + \frac{4\alpha^2\mu_0^2(\xi)G_{12}(\alpha,\xi)G_{11}(\alpha,\xi)}{\left[\lambda_0(\xi)+2\mu_0(\xi)\right]^2} - \right. \\ \left. - \frac{2i\alpha\left[G_{12}(\alpha,\xi)G_{41}(\alpha,\xi) + G_{12}(\alpha,\xi)G_{41}(\alpha,\xi)\right]}{\lambda_0(\xi)+2\mu_0(\xi)} \right\} e^{-i\alpha x_1} d\alpha$$

$$K_{21}(x_1,\xi) = \frac{1}{2\pi}\int_\sigma \Big\{4\alpha^2\mu_0^2(\xi)G_{12}(\alpha,\xi)G_{11}(\alpha,\xi) + G_{42}(\alpha,\xi)G_{41}(\alpha,\xi) + \\ +2i\alpha\mu_0(\xi)\left[G_{12}(\alpha,\xi)G_{41}(\alpha,\xi) + G_{11}(\alpha,\xi)G_{42}(\alpha,\xi)\right]\Big\} \frac{e^{-i\alpha x_1}}{\left[\lambda_0(\xi)+2\mu_0(\xi)\right]^2} d\alpha$$

$$K_{31}(x_1,\xi) = \frac{1}{2\pi}\int_\sigma \left[G_{12}(\alpha,\xi)G_{11}(\alpha,\xi) - G_{22}(\alpha,\xi)G_{21}(\alpha,\xi)\right] e^{-i\alpha x_1} d\alpha$$

$$u_2^1(x_1,1) = -\int_0^1 \Big\{\mu_1(\xi)K_{12}(x_1,\xi) + \left[\lambda_1(\xi) + 2\mu_1(\xi)\right] K_{22}(x_1,\xi) - \\ -\kappa_2^2\rho_1(\xi)K_{32}(x_1,\xi)\Big\} d\xi, \tag{22}$$

$$K_{12}(x_1,\xi) = \\ = \frac{1}{2\pi}\int_\sigma \left\{ \frac{G_{32}^2(\alpha,\xi)}{\mu_0^2(\xi)} + \frac{4\alpha^2\mu_0^2(\xi)G_{12}^2(\alpha,\xi)}{\left[\lambda_0(\xi)+2\mu_0(\xi)\right]^2} - \frac{4i\alpha G_{12}(\alpha,\xi)G_{42}(\alpha,\xi)}{\lambda_0(\xi)+2\mu_0(\xi)} \right\} e^{-i\alpha x_1} d\alpha$$

$$K_{22}(x_1,\xi) = \frac{1}{2\pi}\int_\sigma \Big\{4\alpha^2\mu_0^2(\xi)G_{12}^2(\alpha,\xi) + \\ +G_{42}^2(\alpha,\xi) + 4i\alpha\mu_0(\xi)G_{12}(\alpha,\xi)G_{42}(\alpha,\xi)\Big\} \frac{e^{-i\alpha x_1}}{\left[\lambda_0(\xi)+2\mu_0(\xi)\right]^2} d\alpha$$

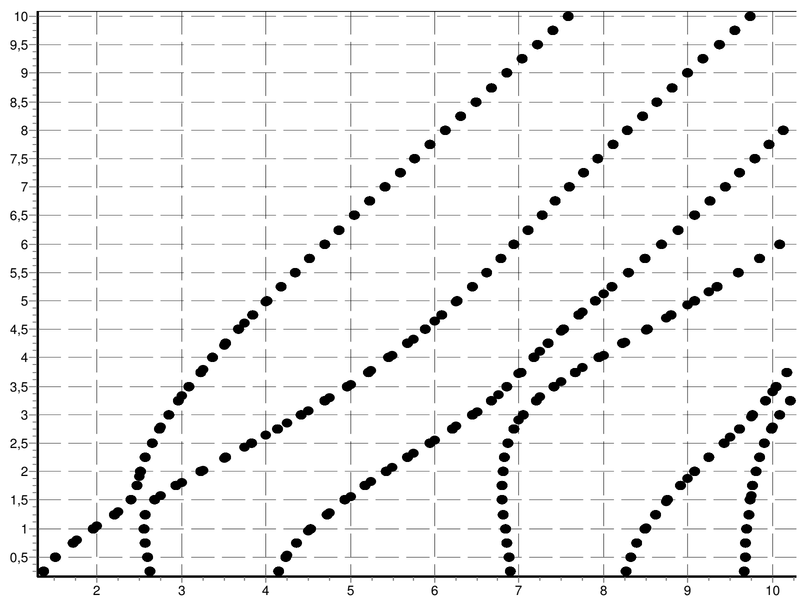

Figure 1: Dispersion curves for $\mu(x) = \mu(0)/(1 + x_2)$, $\rho = \rho(0)e^{0.5x_2}$, $\nu = 0.3$. Horizontal axis corresponds κ_2, vertical axis corresponds α

$$K_{32}(x_1, \xi) = \frac{1}{2\pi} \int\limits_{\sigma} \left[G^2_{12}(\alpha, \xi) - G^2_{22}(\alpha, \xi) \right] e^{-i\alpha x_1} d\alpha$$

Equations (21) and (22) are linear equations, connecting mechanical characteristics corrections with the displacement correction.

Consider equations (21), (22). Assume that $\lambda(x_2)$ and $\lambda(x_2)$ are known and we need to find $\rho(x_2)$ using the vertical displacement data. Then equations (22) turn to:

$$u^1_2(x_1, 1) = \frac{\kappa^2}{2\pi} \int\limits_0^1 \rho_1(\xi) d\xi \int\limits_{\sigma} \left[G^2_{12}(\alpha, \xi) - G^2_{22}(\alpha, \xi) \right] e^{-i\alpha x_1} d\alpha, \tag{23}$$

where f is the observed displacement field and f_0 is the "etalon" displacement field corresponding to distribution $\rho_0(x_2)$.

We choose an initial approximation ρ_0. Solving the problem (6)-(3)-(4) and inverting the Fourier transform, we obtain u^0_2. Then, assuming that u_1 equals the difference between the observed displacement field and the displacement field u_0, we solve (23) and find ρ_1. Then we add the correction ρ_1 to the function ρ_0 and repeat the described procedure until correction becomes negligibly small. The problem of definition of other mechanical parameters can be solved in the similar way.

4 Numerical results

Fig. 1 shows dispersion curves for $\mu(x) = \mu(0)/(1+x_2)$, $\rho = \rho(0)e^{0.5x_2}$, $\nu = 0.3$. Dispersion curves are found from the analysis of integrand denumeratof of (8). The chart depicts values of α and κ_2, for which the integrand denumerator equals zero.

Fig. 2 shows the displacement field for inhomogeneous elastic layer for $\kappa_2 = 3$. Gray line corresponds to the displacement field found using the residual theory [2], [3], and black line corresponds the displacement field found using the direct numerical integration.

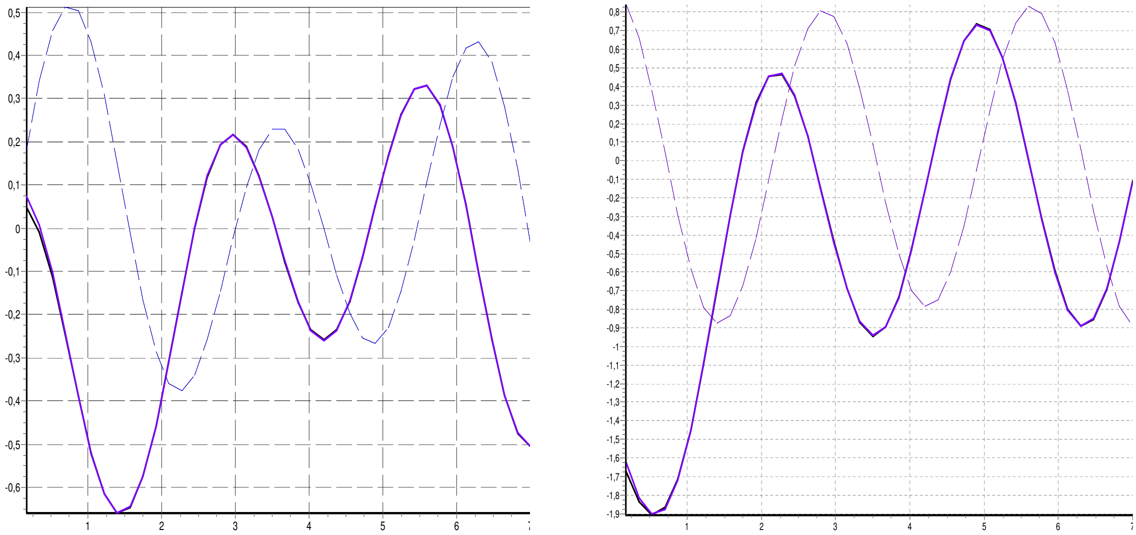

Figure 2: Horizontal (on the left) and vertical (on the right) displacement for $\mu(x) = \mu(0)/(1+x_2)$, $\rho = \rho(0)e^{0.5x_2}$, $\nu = 0.3$. Horizontal axis corresponds x_1, vertical axis corresponds u_1.

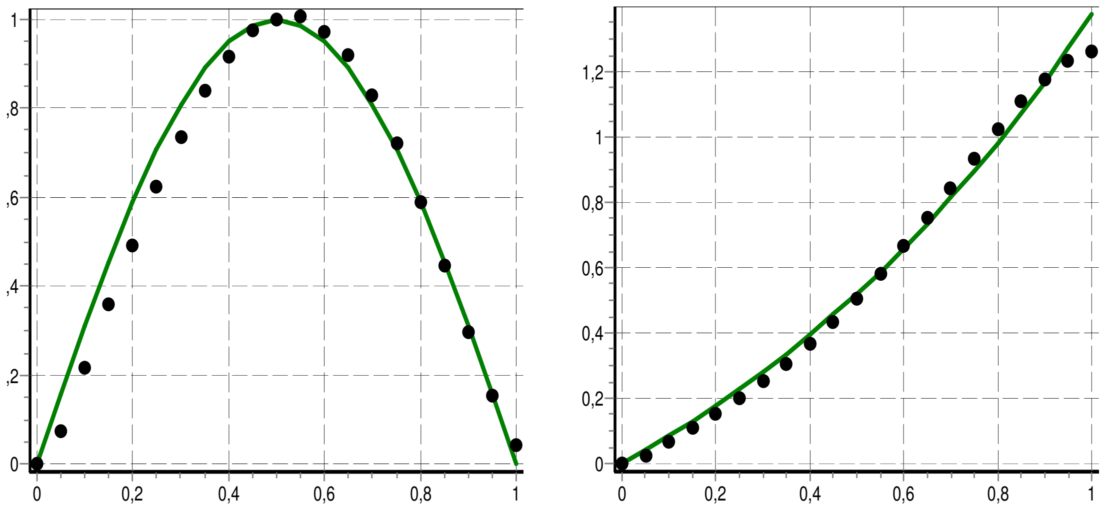

Figure 3: On the left density reconstruction results for $\rho(x_2) = \rho(0)\,(1 + \sin \pi x_2)$ are depicted. 23 iterations were required for reconstruction, On the right - density reconstruction results for $\rho(x_2) = \rho(0)\left[1 + 0.8(e_2^{x} - 1)\right]$. 29 iterations were required

Fig. 3 shows the density reconstruction results for $\kappa_2 = 1$. Horizontal axis corresponds x_2, vertical axis corresponds to the correction to the initial constant approximation. Iterations went on until the correction norm became less than 10^{-4}. Solid lines corresponds the ecact solution, dots correspond reconstructed solution.

Acknowledgements

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References

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