Continuum models with constitutive laws for body forces and moments

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Abstract

There exist the continuum models without any constitutive law, for example: elastic string and membrane, ideal incompressible fluid.

There exist also the models with constitutive law for stresses or for internal surface interactions, for example: elasticity, ideal compressible and viscous fluids, heat conduction.

There exist continuum models with constitutive laws for body forces and body moments, for example: elastic string, beam, membrane, plate, elasticity, hydro- and fluid dynamics, heat conduction where the constitutive laws of the body interactions are added. These models allow to consider the real finite point boundary conditions applied to continuum. That point boundary conditions could be given at any finite point of a continuum or at infinity for unbounded continuum. Some of these models will be presented in the paper. The local and non-local internal body interactions are considered. The boundary conditions are considered in more details because the order of the partial differential equations in the given models is usually higher as in the classical case.

The Galerkin type displacement potential is introduced in case of linear isotropic elasticity with local and non-local internal body forces.

1 Introduction

The statement of the problems in the modern continuum mechanics usually includes the constitutive law for stresses and does not consider the constitutive law for the internal body forces and body moments. The body forces are considered as the external forces like gravitational, electromagnetic forces [5]. Then the linearized theories must accept the solutions with nonphysical singularities in displacements, velocities, temperature, gravitational and electromagnetic fields. The introduction of the internal body forces allows to improve the solutions of the problems at least in the sense of excluding the nonphysical point singularities.

2 Internal body forces and moments

Consider a real solid and let us take some control volume, which includes a fixed number of particles. The control volume is surrounded by a control surface. The particles which are inside the control surface are internal particles and they belong to the control volume. The particles which are outside the control surface are the external particles and they do not belong to control volume. All other particles belong to the boundary particles of the control volume.

There are interactions between particles. The resultant of the forces applied to all internal particles of the control volume from the external particles is the internal body force. The principle moment of the forces and moments applied to all internal forces from the external particles is the internal body moment. The forces and moments applied to the boundary particles of the control volume from the external particles are the surface forces and moments.

Continuum mechanics considers the limit as the control volume tends to zero. Then there are two real possibilities. The first one is when the limit of the control volume will come to a point in an empty space. Then there are no body forces and moments for a small enough volume. The second case is when the limiting point belongs to some particle and the control volume finally consists of one particle inside a control surface and there are no any particles belonging to the control surface. Then there are the body force and body moment and there are no surface forces and moments. It means that the continuum mechanics gives just a mathematical model to the real solid, but it is not unique model.

Continuum mechanics accepts stresses and the constitutive law for stresses. But the constitutive laws for the internal body forces and moments are ignored. There are two other possibilities. The first one is to ignore the stresses and to consider just the internal body forces and moments and the constitutive laws for them. The second possibility is to accept the constitutive law for stresses together with the constitutive laws for the internal body forces and moments.

This paper will use the following local constitutive law for the internal body forces in elastic solid, which does not move as a rigid body:

$$\mathbf{f_{loc}}(\mathbf{r}) = -\alpha_1\mathbf{u} - \alpha_2\nabla^2\mathbf{u} - \alpha_3\nabla^4\mathbf{u}, \tag{1}$$

where $\mathbf{u}$ is the displacement vector, $\alpha_1, \alpha_2, \alpha_3$ are the material constants, which we suppose to be nonnegative, ∇ is the gradient.

The following non-local body force can be taken into account too

$$\mathbf{f_{nloc}}(\mathbf{r}) = \int_V \Big\{\alpha_4[\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{r}')] + \alpha_5\nabla^2[\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{r}')] + \alpha_6\nabla^4[\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{r}')]\Big\}\, d\mathbf{r}', \tag{2}$$

where V is the volume of the body, $\mathbf{r}, \mathbf{r}'$ are the position vectors of the points of the body, $\mathbf{u}(\mathbf{r})$ is the displacement of the point $\mathbf{r}$, $\alpha_1, \alpha_2, \alpha_3$ are the material constants.

The superposition of forces eqs. (1), (2) could be used also.

3 Beam with internal body forces

3.1 Statement of the problem

Consider an elastic beam [8]. The equation of motion in a plane of the beam with internal body forces and body moments could be written in the form

$$T\frac{\partial^2 u}{\partial x^2} + \frac{\partial Q}{\partial x} + f + q(x,t) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{3}$$

$$\frac{\partial T}{\partial x} = 0,\ M = -EI\frac{\partial^2 u}{\partial x^2},\ \theta = \frac{\partial u}{\partial x}, \tag{4}$$

$$f = -\alpha_1 u - \alpha_2\frac{\partial^2 u}{\partial x^2} - \alpha_3\frac{\partial^4 u}{\partial x^4},\ m = -\alpha_4\theta - \alpha_5\frac{\partial^2 \theta}{\partial x^2} - \alpha_6\frac{\partial^4 \theta}{\partial x^4}, \tag{5}$$

where EI is the bending stiffness of the beam, $x-$ is the Cartesian coordinate of a cross-section, $0 \le x \le L$, $L-$ is the length of the beam, $\rho-$ is the density of mass per unit length, $u-$ is the transversal displacement of a cross-section, $t-$ is time, $q(x,t)-$ is the density of the transversal external body forces per unit length, $f-$ is the density of the transversal internal body forces per unit length, $m-$ is the density of the internal body moments per unit length, $M-$ is the bending moment, $Q-$ is the transversal shear force, $T-$ is the tension, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6-$ are materials constants.

Let us substitute the eqs. (4), (5) and into the eq. (3). Then the following equation will be obtained

$$\alpha_6 \frac{\partial^6 u}{\partial x^6} - (EI + \alpha_3 - \alpha_5) \frac{\partial^4 u}{\partial x^4} + (T + \alpha_4 - \alpha_2) \frac{\partial^2 u}{\partial x^2} - \alpha_1 u + q(x,t) = \rho \frac{\partial^2 u}{\partial t^2}. \tag{6}$$

The boundary conditions will be discussed in the next section. The initial conditions are given displacement u and its time derivative $\frac{\partial u}{\partial t}$ given at an initial time t_0.

3.2 Principle of virtual work and boundary conditions

Consider the steady state problem given by the eq. (6)

$$\alpha_6 \frac{\partial^6 u}{\partial x^6} - (EI + \alpha_3 - \alpha_5) \frac{\partial^4 u}{\partial x^4} + (T + \alpha_4 - \alpha_2) \frac{\partial^2 u}{\partial x^2} - \alpha_1 u + q(x,t) = 0. \tag{7}$$

Let $\delta u(x)$ is a virtual displacement of the beam. Then multiplying the Eq. (7) by δu and integrating the result from 0 to L we will get

$$\int_0^L \left(\alpha_6 \frac{\partial^6 u}{\partial x^6} - (EI + \alpha_3 - \alpha_5) \frac{\partial^4 u}{\partial x^4} + (T + \alpha_4 - \alpha_2) \frac{\partial^2 u}{\partial x^2} - \alpha_1 u + q \right) \delta u(x) dx = 0. \tag{8}$$

We can use the integration by parts and transform the eq. (8) to the form

$$\int_0^L \frac{1}{2} \delta \left[(T + \alpha_4 - \alpha_2) u'^2 + \alpha_1 u^2 + (EI + \alpha_3 - \alpha_5) u''^2 + \alpha_6 u'''^2 \right] dx = \tag{9}$$

$$= \int_0^L q \delta u dx + \left[\alpha_6 u''''' + (T + \alpha_4 - \alpha_2) u' - (EI + \alpha_3 - \alpha_5) u''' \right] \delta u|_0^L + \tag{10}$$

$$+ \left[(EI + \alpha_3 - \alpha_5) u'' - \alpha_6 u'''' \right] \delta u'|_0^L + \alpha_6 u''' \delta u''|_0^L. \tag{11}$$

The following boundary conditions at the ends of the beam could be applied:

- either $P = Tu' + Q$ or u is prescribed,
- either $M = EIu''$ or u' is prescribed,
- $EIu'''' - Tu'' = 0$.

The first two conditions are traditional for a beam problems. The last condition is obtained from eq. (7) where all terms corresponding to any body forces or moments are excluded.

The external transversal forces at the ends of a beam should be calculated according to the eqs.

$$P(L) = Q(L) + Tu'(L), P(0) = -Q(0) - Tu'(0), \tag{12}$$

$$M(L) = -EIu''(L), M(0) = EIu''(0). \tag{13}$$

If we accept the existence of the potential energy and the potential energy of a beam is introduced in the form

$$\Pi = \frac{1}{2}\int_0^L \left[(T+\alpha_4-\alpha_2)u'^2+\alpha_1 u^2+(EI+\alpha_3-\alpha_5)u''^2+\alpha_6 u'''^2\right] dx \tag{14}$$

then the principle of virtual work could be applied:

$$\delta\Pi = \delta' W, \tag{15}$$

where the variation of the potential energy is

$$\delta\Pi = \frac{1}{2}\delta\int_0^L \left[(T+\alpha_4-\alpha_2)u'^2+\alpha_1 u^2+(EI+\alpha_3-\alpha_5)u''^2+\alpha_6 u'''^2\right] dx. \tag{16}$$

The virtual work of the external forces q, P and moments M is

$$\delta' W = \int_0^L q\delta u dx + P\delta u|_0^L - M\delta u'|_0^L, \tag{17}$$

where P, M are the transversal force and bending moment applied at the ends of the beam. If we substitute the eqs. (16), (17) into the eq. (15) then the following equation will be available

$$\frac{1}{2}\delta\int_0^L \left[(T+\alpha_4-\alpha_2)u'^2+\alpha_1 u^2+(EI+\alpha_3-\alpha_5)u''^2+\alpha_6 u'''^2\right] dx = \tag{18}$$

$$= \int_0^L q\delta u dx + P\delta u|_0^L - M\delta u'|_0^L. \tag{19}$$

Subtracting eq. (18), (19) from the eq. (9), (10), (11) we will get

$$\left[\alpha_6 u''''' + (T+\alpha_4-\alpha_2)u' - (EI+\alpha_3-\alpha_5)u''' - P\right]\delta u|_0^L + \tag{20}$$

$$+\left[(EI+\alpha_3-\alpha_5)u'' - \alpha_6 u'''' + M\right]\delta u'|_0^L + \alpha_6 u'''\delta u''|_0^L \tag{21}$$

The eq. (20), (21) shows that the following boundary conditions at the ends of the beam could be applied

- either $M = (EI+\alpha_3-\alpha_5)u'' - \alpha_6 u''''$ or u' is prescribed,
- either $P = \alpha_6 u''''' + (T+\alpha_4-\alpha_2)u' - (EI+\alpha_3-\alpha_5)u'''$ or u is prescribed,
- either $u''' = 0$ or u'' is prescribed.

3.3 Example of beam with internal body forces and moments

If we consider the beam problem with internal body forces, where $\alpha_5 = 0, \alpha_6 = 0$ then the differential equation of the problem will take the form

$$(EI+\alpha_3)\frac{d^4u}{dx^4} + (\alpha_2-\alpha_4)\frac{d^2u}{dx^2} + \alpha_1 u = 0. \tag{22}$$

The boundary conditions are taken the equations

$$u(0) = u_0 \neq 0, u(\infty) = 0, \tag{23}$$

$$\frac{d^2u}{dx^2}(0) = 0, \frac{d^2u}{dx^2}(\infty) = 0. \tag{24}$$

The solution of the problem with internal body forces Eqs. (22), (23), (24) exists and equals

$$u = \frac{u_0}{\lambda_2^2 - \lambda_1^2}(\lambda_2^2 \exp \lambda_1 x - \lambda_1^2 \exp \lambda_2 x), \tag{25}$$

where

$$\lambda_{1,2} = -\sqrt{\frac{\alpha_4 - \alpha_2 \pm \sqrt{(\alpha_4 - \alpha_2)^2 - 4\alpha_1(\alpha_3 + EI)}}{2(EI + \alpha_3)}}. \tag{26}$$

The considered example of the beam problem shows that the introduced internal body forces and the internal body moments allow to obtain solutions in the cases, where the classical problem does not have any solution.

4 Membrane with internal body forces

4.1 Statement of the problem

Let us consider an elastic membrane. The equation of motion of the membrane is described in [2], [10], [13], [14], [16], [17] and [18]. This membrane equation with internal body forces is

$$T_0\nabla^2 u - \alpha_1 u - \alpha_2\nabla^2 u - \alpha_3\nabla^4 u + q(x, y, t) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{27}$$

where the membrane lies in the plane (x, y) in its natural state, T_0 is its tension per a unit of length, $u(x, y, t)$ is the transversal displacement of the point (x, y) of the initially plane membrane, which occupies the plane domain Ω with the boundary Γ, ρ is the density of mass per unit area, t is time, $q(x, y, t)$ is the density of the transversal external body forces per unit area. The tension T_0 is constant in this statement of the problem. The internal transversal body forces are taken in the following form

$$f = -\alpha_1 u - \alpha_2\nabla^2 u - \alpha_3\nabla^4 u. \tag{28}$$

4.2 Principle of virtual work and boundary conditions

Consider the steady state membrane problem given by the eq. (27)

$$T_0\nabla^2 u - \alpha_1 u - \alpha_2\nabla^2 u - \alpha_3\nabla^4 u + q = 0. \tag{29}$$

Let $\delta u(x)$ is a virtual displacement of the membrane and P is the transversal force per unit length acting at the end points of the membrane. Then multiplying the Eq. (29) by δu and integrating the result over the domain occupied by the membrane Ω we will get

$$\int_\Omega \left(T_0\nabla^2 u - \alpha_1 u - \alpha_2\nabla^2 u - \alpha_3\nabla^4 u + q\right)\delta u d\Omega = 0. \tag{30}$$

We can use the integration by parts and the first Green formula to transform the integral eq. (30) to the form

$$\frac{1}{2}\delta\int_\Omega \left[\alpha_3\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right) + (T_0 - \alpha_2)\left(u_x^2 + u_y^2\right) + \alpha_1 u^2 - 2q\right] d\Omega + \tag{31}$$

$$+\oint_\Gamma \left\{ \left[\alpha_3 \nabla^2 u_\nu - (T_0 - \alpha_2) u_\nu \right] \delta u - \alpha_3 u_{\nu\nu} \delta u_\nu - \alpha_3 u_{s\nu} \delta u_s \right\} d\Gamma = 0, \tag{32}$$

where The external transversal forces per unit length at the boundary of a membrane should be calculated according to the equation

$$P = T_0 u_\nu. \tag{33}$$

If we accept the existence of the potential energy and the potential energy of a membrane is introduced in the form

$$\Pi = \frac{1}{2} \int_\Omega \left[\alpha_3 \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \right) + (T_0 - \alpha_2) \left(u_x^2 + u_y^2 \right) + \alpha_1 u^2 \right] d\Omega \tag{34}$$

then the principle of virtual work could be applied:

$$\delta \Pi = \delta' W, \tag{35}$$

where the variation of the potential energy is

$$\delta \Pi = \frac{1}{2} \delta \int_\Omega \left[\alpha_3 \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \right) + (T_0 - \alpha_2) \left(u_x^2 + u_y^2 \right) + \alpha_1 u^2 \right] d\Omega. \tag{36}$$

The virtual work of the external forces q, P is

$$\delta' W = \int_\Omega q \delta u dx + \oint_\Gamma P \delta u d\Gamma, \tag{37}$$

where P is the transversal force per unit length applied at the boundary of the membrane. If we substitute the eqs. (36), (37) into the eq. (35) then the following equation will be available

$$\frac{1}{2} \delta \int_\Omega \left[\alpha_3 \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \right) + (T_0 - \alpha_2) \left(u_x^2 + u_y^2 \right) + \alpha_1 u^2 \right] d\Omega = \int_\Omega q \delta u dx + \oint_\Gamma P \delta u d\Gamma. \tag{38}$$

Subtracting eq. (38) from the eq. (31), (32) we will get

$$\oint_\Gamma \left\{ \left[-\alpha_3 \nabla^2 u_\nu + (T_0 - \alpha_2) u_\nu - P \right] \delta u + \alpha_3 u_{\nu\nu} \delta u_\nu + \alpha_3 u_{s\nu} \delta u_s \right\} d\Gamma = 0. \tag{39}$$

The eq. (39) shows that the following boundary conditions could be applied to the membrane

- either $u_{\nu\nu} = 0$ or u_ν is prescribed,
- either $P = -\alpha_3 \nabla^2 u_\nu + (T_0 - \alpha_2) u_\nu$ or u is prescribed,
- either $u_{s\nu} = 0$ or u_s is prescribed.

4.3 Example of membrane with internal body forces

If we consider the steady state membrane problem with the internal body forces and $\alpha_3 \neq 0$, then the differential equation of the problem will take the form

$$T_0\nabla^2 u - \alpha_1 u - \alpha_2\nabla^2 u - \alpha_3\nabla^4 u = 0. \tag{40}$$

We are looking for a solution of the Eq. (40) $u = u(r)$, which satisfies the boundary conditions

$$u(0) = u_0 \neq 0, u(\infty) = 0. \tag{41}$$

We will see that these boundary conditions Eq. (41) are sufficient to obtain the solution of the given problem. The Eq. (40) could be written in the following form

$$(\nabla^2 - s_1^2)(\nabla^2 - s_2^2)u = 0, \tag{42}$$

where

$$s_1 = |\lambda_1|, s_2 = |\lambda_2|, \tag{43}$$

and λ_1, λ_2 are given according to

$$\lambda_{1,2} = -\sqrt{\frac{T_0 - \alpha_2 \pm \sqrt{(T_0 - \alpha_2)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}}. \tag{44}$$

The general solution of the Eq. (42) is the sum of two functions:

$$u = u_1 + u_2, \tag{45}$$

where u_1, u_2 are the general solutions of the equations

$$\nabla^2 u_i - s_i^2 u_i = 0,\ i = 1, 2. \tag{46}$$

The Eqs. (46) are the Bessel equations. Then the general solution of the Eq. (42) will be

$$u = C_1 I_0(s_1 r) + C_2 K_0(s_1 r) + C_3 I_0(s_2 r) + C_4 K_0(s_2 r), \tag{47}$$

where I_0, K_0 are the Macdonald functions and C_1, C_2, C_3, C_4 are the arbitrary constants. The bounded solution of stated problem is

$$u(r) = \frac{u_0}{\ln\frac{s_2}{s_1}}\left[K_0(s_1 r) - K_0(s_2 r)\right]. \tag{48}$$

This example shows that the solution does not have a singularity at the origin and at the infinity and that corresponds to the real situation with real membrane. And that is impossible in the classical theory.

5 Elasticity with internal body forces

5.1 Statement of the problem

There are many books, where the different elasticity problems are taken into consideration, for example [1], [3], [4], [5], [6], [8], [11], [15] and many other papers and manuscripts.The differential equations of the stated problem are the equations of the linear isotropic elasticity in 3D domain [9]. We have

$$\varrho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \varrho_0 \mathbf{B} + (\lambda + \mu)\nabla e + \mu \nabla^2 \mathbf{u}, \tag{49}$$

where dilatation e equals

$$e = \mathtt{div}\mathbf{u} \tag{50}$$

and $\mathbf{u}$ is the displacement vector, ϱ_0 is the density, $\varrho_0\mathbf{B}$ is the external body force per unit volume, λ and μ are Lame's coefficients or Lame's constants, ∇ is the gradient, ∇^2 is the Laplacian.

Let us consider a linear isotropic elastic body with internal body forces. The governing equations are taken in case of the steady state problem

$$\nabla \mathrm{div}\mathbf{u} + (1 - 2\nu)\nabla^2 \mathbf{u} - \alpha_1 \mathbf{u} - \alpha_2 \nabla^2 \mathbf{u} - \alpha_3 \nabla^4 \mathbf{u} + \varrho_0 \mathbf{B} = 0, \tag{51}$$

where ν is the Poisson ratio and the internal body force is taken in the form of the Eq. (1).

The eq. (51) could be written in the form

$$\sigma_{ij,j} - \alpha_1 u_i - \alpha_2 \nabla^2 u_i - \alpha_3 \nabla^4 u_i + \varrho_0 B_i = 0, \tag{52}$$

where σ_{ij} are the components of the stress tensor or stresses, $i, j = 1, 2, 3$.

The differential Eqs. (51) have the fourth order therefore the second boundary condition should be given at the boundary surface with respect to the classical case. It seems to be possible to apply the following boundary conditions at the boundary surface: given

- displacements and Eq. (51) without internal body forces
- stresses and Eq. (51) without internal body forces
- displacements and stresses
- displacements and stresses as a function of displacements
- stresses and displacements as a function of stresses
- stresses as a function of displacements and Eq. (51) without internal body forces
- conditions obtained in the variational formulation of the problem
- other possible conditions.

5.2 Principle of virtual work and boundary conditions

The variational description of the elasticity problems could be found in many books. We are using [7] and [19]. Consider the steady state problem given by the eq. (52)

$$\sigma_{ij,j} - \alpha_1 u_i - \alpha_2 \nabla^2 u_i - \alpha_3 \nabla^4 u_i + \varrho_0 B_i = 0. \tag{53}$$

Let $\delta u_i(x)$ is a virtual displacement of the solid. Then multiplying the Eq. (53) by δu_i and integrating the result over the domain Ω we will get

$$\int_\Omega \Big(\sigma_{ij,j} - \alpha_1 u_i - \alpha_2 \nabla^2 u_i - \alpha_3 \nabla^4 u_i + \varrho_0 B_i\Big)\, \delta u_i d\Omega = 0. \tag{54}$$

We can use the integration by parts and transform the following integral in eq. (54) to the form

$$\int_\Omega \sigma_{ij}\delta e_{ij} d\Omega + \int_\Omega \alpha_1 u_i \delta u_i d\Omega + \int_\Omega \alpha_2 u_{i,k}\delta u_{i,k} d\Omega + \tag{55}$$

$$+\int_\Omega \alpha_3 u_{i,kj}\delta u_{i,kj} d\Omega = \int_S \sigma_{ij}\delta u_i \nu_j dS + \int_S \alpha_2 u_{i,j}\nu_j \delta u_i dS + \tag{56}$$

$$+\int_\Omega \varrho_0 B_i \delta u_i d\Omega + \int_S \alpha_3 u_{i,kj}\nu_j \delta u_{i,k} dS - \int_S \alpha_3 \nabla^2(u_{i,j})\nu_j \delta u_i dS. \tag{57}$$

The eq. (55), (56), (57) represents the variational statement of the given problem. The traction at the boundary surface should be calculated as

$$T_{\nu i} = \sigma_{ij}\nu_j. \tag{58}$$

If the strain-energy function $W(e_{ij})$ is introduced, then the eq. (55), (56), (57) will be transformed to the form

$$\delta \int_\Omega (W + \frac{1}{2}\alpha_1 u_i u_i + \frac{1}{2}\alpha_2 u_{i,k} u_{i,k} + \frac{1}{2}\alpha_3 u_{i,kj} u_{i,kj}) d\Omega = \tag{59}$$

$$= \int_S \left[T_{\nu i} + \alpha_2 u_{i,j}\nu_j - \alpha_3 \nabla^2(u_{i,j})\nu_j\right] \delta u_i dS + \tag{60}$$

$$+\int_S \alpha_3 u_{i,kj}\nu_j \delta u_{i,k} dS + \int_\Omega \varrho_0 B_i \delta u_i d\Omega. \tag{61}$$

If the potential energy of the solid is accepted in the form

$$\Pi = \int_\Omega (W + \frac{1}{2}\alpha_1 u_i u_i + \frac{1}{2}\alpha_2 u_{i,k} u_{i,k} + \frac{1}{2}\alpha_3 u_{i,kj} u_{i,kj}) d\Omega, \tag{62}$$

then the principle of virtual work applied to the given solid is

$$\delta\Pi = \int_S T_{\nu i}\delta u_i dS + \int_\Omega \varrho_0 B_i \delta u_i d\Omega, \tag{63}$$

where $T_{\nu i}$ are the components of the applied at the boundary surface traction.

Subtract eq. (63) from eq. (55), (56), (57) to get

$$\int_S \left[\sigma_{ij}\nu_j + \alpha_2 u_{i,j}\nu_j - \alpha_3 \nabla^2(u_{i,j})\nu_j - T_i\right] \delta u_i dS + \int_S \alpha_3 u_{i,kj}\nu_j \delta u_{i,k} dS = 0. \tag{64}$$

The eq. (64) shows that the following boundary conditions could be applied

- either $T_i = \sigma_{ij}\nu_j + \alpha_2 u_{i,j}\nu_j - \alpha_3 \nabla^2(u_{i,j})\nu_j$ or u_i are prescribed,
- either $\alpha_3 u_{i,kj}\nu_j = 0$ or $u_{i,k}$ are prescribed.

5.3 Example of elasticity without internal body forces

Let us take an example of classical linear isotropic elastic problem considered in more details in [12]. An elastic body occupies the unbounded cylinder $0 \le r \le R$, where R is the finite radius of the cylinder. Let the displacement field of the body is in cylindrical coordinates r, φ, z:

$$u_r = u_r(r,\varphi),\ u_\varphi = u_\varphi(r,\varphi),\ u_z = u_z(r). \tag{65}$$

Then the component u_z satisfies the equation

$$\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} = 0. \tag{66}$$

This equation for u_z can be considered separately from the equations for the components u_r, u_φ if the boundary conditions allow that. We can suppose for simplicity that $u_r \equiv 0, u_\varphi \equiv 0$. Let us apply the boundary conditions for u_z

$$u_z(0) = u_0 \neq 0,\ u_z(R) = 0. \tag{67}$$

The problem Eqs. (66), (67) coincides with the classical membrane problem and the conclusions obtained in membrane problem should be repeated here: a continuous solution of the stated problem does not exist for any finite or infinite radius of the cylinder.

5.4 Example of elasticity with internal body forces

Consider now the same problem as in the previous example but the internal body force has all three nonzero coefficients. The Eq. (51) is taken into consideration. The infinite cylinder of radius R is considered. The radius $R = \infty$ for simplicity. The distribution of displacements is given

$$u_r = 0, u_\varphi = 0, u_z = u_z(r). \tag{68}$$

Then the differential equation for u_z will take the form

$$\alpha_3 \nabla^4 u_z + (\alpha_2 - \mu)\nabla^2 u_z + \alpha_1 u_z = 0, \tag{69}$$

where operator ∇ has the following expression

$$\nabla = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{70}$$

The solution of the stated problem could be taken from correspondent solution of the similar membrane problem and it will be in the considered case

$$u_z(r) = \frac{u_0}{\ln \frac{s_2}{s_1}}\left[K_0(s_1 r) - K_0(s_2 r)\right], \tag{71}$$

where

$$s_{1,2} = \sqrt{\frac{\mu - \alpha_2 \pm \sqrt{(\mu-\alpha_2)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}}, \tag{72}$$

The solution Eq. (71) satisfies also the conditions

$$\frac{du_z}{dr}(0) = 0, \frac{du_z}{dr}(\infty) = 0. \tag{73}$$

5.5 Galerkin type of displacement potential

The solution of the Eq. (51) can be obtained using similar methods as in the classical theory without internal body forces. Consider for example the displacement potentials method following [5].

Consider the equation (51), where $\alpha_2 = 0$

$$\nabla \mathrm{div}\mathbf{u} + (1-2\nu)\nabla^2\mathbf{u} - \alpha_1\mathbf{u} - \alpha_3\nabla^4\mathbf{u} = 0. \tag{74}$$

The vector displacement potential $\mathbf{F}$ is introduced in the form

$$2\mu\mathbf{u} = \left[2(1-\nu)\nabla^2 - \nabla\mathrm{div} - \alpha_1 - \alpha_3\nabla^4\right]\mathbf{F}. \tag{75}$$

If the expression for $\mathbf{u}$ in Eq. (75) is substituted into the Eq. (74) then the following differential equation with respect to the potential $\mathbf{F}$ will be obtained

$$(\nabla^2 - s_1)(\nabla^2 - s_2)(\nabla^2 - s_3)(\nabla^2 - s_4)\mathbf{F} = 0, \tag{76}$$

where

$$s_{1,2} = \frac{1-\nu \pm \sqrt{(1-\nu)^2 - \alpha_1\alpha_3}}{\alpha_3},\ s_{3,4} = \frac{1-2\nu \pm \sqrt{(1-2\nu)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}. \tag{77}$$

Then $\mathbf{F}$ could be presented as the sum of four functions

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \mathbf{F_4}, \tag{78}$$

where the functions $\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}$ satisfy the equations

$$\left(\nabla^2 - s_i\right)\mathbf{F_i} = 0,\ i = 1,2,3,4. \tag{79}$$

If the non-local body forces are taken then the constant should be added into the solution (78) and the functions $\mathbf{F_i}$ must satisfy the equation which will be obtained substituting the solution (78) with constant into the governing equation.

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