New cases of integrability in multidimensional dynamics in a nonconservative field

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Abstract

Study of the dynamics of a multidimensional solid depends on the force-field structure. As reference results, we consider the equations of motion of low-dimensional solids in the field of a medium-drag force. Then it becomes possible to generalize the dynamic part of equations to the case of the motion of a solid, which is multidimensional in a similarly constructed force field, and to obtain the full list of transcendental first integrals. The obtained results are of importance in the sense that there is a nonconservative moment in the system, whereas it is the potential force field that was used previously.

1 Introduction

We study nonconservative systems for which the methods for studying, for example, Hamiltonian systems is not applicable in general. Therefore, for such systems, it is necessary, in some sense, to "directly" integrate the main equation of dynamics. We generalize old cases and also obtain new cases of complete integrability in transcendental functions in two-, three-, and four-dimensional rigid body dynamics in a nonconservative force field.

We obtain a whole spectrum of complete integrability cases for nonconservative dynamical systems having nontrivial symmetries. Moreover, in almost all cases, each of the first integrals is expressed through a finite combination of elementary functions, being one transcendental function of its variables. In this case, the transcendence is understood in the complex analysis sense, when after continuation of given functions to the complex domain, they have essentially singular points. The latter fact is stipulated by the existence of attracting and repelling sets in the system (for example, attracting and repelling foci) [1].

We introduce the class of autonomous dynamic systems having one periodic phase coordinate, and therefore, possessing the certain symmetries which are typical for the pendulum-like systems. We show that offered class of systems are embedded to the class of zero mean variable dissipation systems by natural way, i.e., on the average, for the period of the existing periodic coordinate, the sop and diffusing to energy balance to each other in certain sense. We offer the examples of pendulum-like systems on lower-dimension manifolds from dynamics of a rigid body in a nonconservative field of force [1, 2, 3, 4].

In [1, 5] the obtained results are systematized on study of the dynamic equations of the motion of symmetrical three-dimensional ($3D-$) rigid body which residing in a certain nonconservative field of the forces. Its type is also unoriginal from dynamics of the real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it either forces the value of the velocity of a certain typical point of the rigid body to remain as constant in all time of motion, that means the presence in system of nonintegrable servo-constraint.

Therefore, in [5] three additional transcendental first integrals expressing through the finite combination of elementary functions are found to having analytical invariant relations (nonintegrable constraint and the integral on the equality to zero one of the component of angular velocity).

The question on tensor of inertia of four-dimensional ($4D-$) rigid body is considered. It is proposed to study two possible cases logically on principal moments of inertia, i.e., when there exists *two* relations on the principal moments of inertia: (i) when there exist *three* equal principal moments of inertia ($I_2 = I_3 = I_4$); (ii) when there exist *two pairs* of equal moments of inetria ($I_1 = I_2, I_3 = I_4$).

In this activity the obtained results are systematized on study of the dynamic equations of the motion of symmetrical four-dimensional ($4D-$) rigid body which residing in a certain nonconservative field of the forces for the case (ii). Its type is also unoriginal from dynamics of lower-dimensional real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it forces both the value of the velocity of a certain typical point of the rigid body and the certain phase variable to remain as constant in all time, that means the presence in system of nonintegrable servo-constraints.

Many results of this work were regularly reported at numerous workshops, including the workshop "Actual Problems of Geometry and Mechanics" named after professor V. V. Trofimov led by D. V. Georgievskii and M. V. Shamolin.

2 Cases of integrability corresponding to a rigid body motion in four-dimensional space

2.1 More general problem of the motion with the tracing force

Let consider the motion of a homogeneous dynamically symmetric rigid body with "the front end-wall" (two-dimensional disk interacting with a medium which filling the four-dimensional space) in the field of force $\mathbf{S}$ of the resistance under the conditions of quasistationarity.

Let $(v, \alpha, \beta_2, \beta_1)$ are the coordinates of the vector velocity of a certain typical point D of a rigid body (D is the center of two-dimensional disk) such that α is the angle between the vector $\mathbf{v}_D$ and the plane Dx_1x_2, β_2 is the angle measured in the plane Dx_1x_2 up to the projection of the vector $\mathbf{v}_D$ on the plane Dx_1x_2, β_1 is the angle mesured in the plane Dx_3x_4 up to the projection of the vector $\mathbf{v}_D$ on the plane Dx_3x_4,

$$\Omega = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

is the angular velocity tensor of the body, $Dx_1x_2x_3x_4$ is the coordinate system related to the body, herewith, the straight line CD lies in the plane Dx_1x_2 (C is the center of mass), and the axes Dx_3, Dx_4 lie in the disk plane, $I_1,\ I_2 = I_1,\ I_3,\ I_4 = I_3,\ m$ are the inertia–mass characteristics.

Let accept the following decompositions in the projections on the axes of the coordinate system $Dx_1x_2x_3x_4$:

$$\mathbf{DC} = \{\sigma \sin\gamma, -\sigma\cos\gamma, 0, 0\},$$

$$\mathbf{v}_D = \{v\cos\alpha\sin\beta_2, v\cos\alpha\cos\beta_2, v\sin\alpha\cos\beta_1, v\sin\alpha\sin\beta_1\}. \quad (1)$$

Herewith, in our case the decomposition will be also correct for the function of a medium interaction on four-dimensional body: $\mathbf{S} = \{S_1, S_2, 0, 0\}$, $S_1 = S\sin\gamma$, $S_2 = -S\cos\gamma$, $\gamma =$ const, i.e. in this case $\mathbf{F} = \mathbf{S}$, and the angle γ is measured in the plane Dx_1x_2.

Then those part of dynamic equations of the body motion (including and in the case of Chaplygin analytical functions, see below) which describes the center of mass motion and corresponds to the space $\mathbf{R}^4$ under which the tangent forces to three-dimensional disk are absent, has the form:

$$\dot{v}\cos\alpha\sin\beta_2 - \dot{\alpha}v\sin\alpha\sin\beta_2 + \dot{\beta}_2 v\cos\alpha\cos\beta_2 -$$
$$-\omega_6 v\cos\alpha\cos\beta_2 + \omega_5 v\sin\alpha\cos\beta_1 - \omega_3 v\sin\alpha\sin\beta_1 -$$
$$-\sigma(\omega_6^2 + \omega_5^2 + \omega_3^2)\sin\gamma - \sigma(\omega_4\omega_5 + \omega_2\omega_3)\cos\gamma + \sigma\dot{\omega}_6\cos\gamma = \frac{S_1}{m}, \quad (2)$$

$$\dot{v}\cos\alpha\cos\beta_2 - \dot{\alpha}v\sin\alpha\cos\beta_2 - \dot{\beta}_2 v\cos\alpha\sin\beta_2 +$$
$$+\omega_6 v\cos\alpha\sin\beta_2 - \omega_4 v\sin\alpha\cos\beta_1 + \omega_2 v\sin\alpha\sin\beta_1 +$$
$$+\sigma(\omega_6^2 + \omega_4^2 + \omega_2^2)\cos\gamma + \sigma(\omega_4\omega_5 + \omega_2\omega_3)\sin\gamma + \sigma\dot{\omega}_6\sin\gamma = \frac{S_2}{m}, \quad (3)$$

$$\dot{v}\sin\alpha\cos\beta_1 + \dot{\alpha}v\cos\alpha\cos\beta_1 - \dot{\beta}_1 v\sin\alpha\sin\beta_1 -$$
$$-\omega_5 v\cos\alpha\sin\beta_2 + \omega_4 v\cos\alpha\cos\beta_2 - \omega_1 v\sin\alpha\sin\beta_1 +$$
$$+\sigma(\omega_4\omega_6 - \omega_1\omega_3)\sin\gamma - \sigma(\omega_5\omega_6 + \omega_1\omega_2)\cos\gamma - \sigma\dot{\omega}_5\sin\gamma - \sigma\dot{\omega}_4\cos\gamma = 0, \quad (4)$$

$$\dot{v}\sin\alpha\sin\beta_1 + \dot{\alpha}v\cos\alpha\sin\beta_1 + \dot{\beta}_1 v\sin\alpha\cos\beta_1 +$$
$$+\omega_3 v\cos\alpha\sin\beta_2 - \omega_2 v\cos\alpha\cos\beta_2 + \omega_1 v\sin\alpha\cos\beta_1 -$$
$$-\sigma(\omega_2\omega_6 + \omega_1\omega_5)\sin\gamma + \sigma(\omega_3\omega_6 - \omega_1\omega_4)\cos\gamma + \sigma\dot{\omega}_3\sin\gamma + \sigma\dot{\omega}_2\cos\gamma = 0, \quad (5)$$

where $S = s(\alpha)v^2$, $\sigma = CD$, $v > 0$.

Those part of the dynamic equations of the body motion which describes the body motion around the center of mass, and corresponds to the Lie algebra so(4), has the form:

$$(\lambda_4 + \lambda_3)\dot{\omega}_1 + (\lambda_3 - \lambda_4)(\omega_3\omega_5 + \omega_2\omega_4) = 0, \quad (6)$$

$$(\lambda_2 + \lambda_4)\dot{\omega}_2 + (\lambda_2 - \lambda_4)(\omega_3\omega_6 - \omega_1\omega_4) = -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\cos\gamma, \quad (7)$$

$$(\lambda_4 + \lambda_1)\dot{\omega}_3 + (\lambda_4 - \lambda_1)(\omega_2\omega_6 + \omega_1\omega_5) = -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\sin\gamma, \quad (8)$$

$$(\lambda_3 + \lambda_2)\dot{\omega}_4 + (\lambda_2 - \lambda_3)(\omega_5\omega_6 + \omega_1\omega_2) = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\cos\gamma, \quad (9)$$

$$(\lambda_1 + \lambda_3)\dot{\omega}_5 + (\lambda_3 - \lambda_1)(\omega_4\omega_6 - \omega_1\omega_3) = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\sin\gamma, \quad (10)$$

$$(\lambda_1 + \lambda_2)\dot{\omega}_6 + (\lambda_1 - \lambda_2)(\omega_4\omega_5 + \omega_2\omega_3) = 0. \quad (11)$$

Thus, the following direct product of four-dimensional manifold on the Lie algebra so(4) is the phase space of the tenth order system (2)–(5), (6)–(11): $\mathbf{R}^1 \times \mathbf{S}^3 \times \mathrm{so}(4)$.

We notice right now that the system (2)–(5), (6)–(11), by virtue of the having dynamical symmetry

$$I_1 = I_2,\ I_3 = I_4, \tag{12}$$

possesses the cyclic first integrals

$$\omega_1 \equiv \omega_1^0 = \text{const},\ \omega_6 \equiv \omega_6^0 = \text{const}. \tag{13}$$

Herewith, hereinafter we shall consider the dynamics of the system on zero level:

$$\omega_1^0 = \omega_6^0 = 0. \tag{14}$$

And if there exists the more general problem of the body motion with the certain tracing force $\mathbf{T}$, which acts on the plane Dx_1x_2 and providing the fulfillment of the following equalities in all time of the motion

$$v \equiv \text{const},\ \beta_2 \equiv \text{const}, \tag{15}$$

that in the system (2)–(5), (6)–(11) the values $T_1 + S_1,\ T_2 + S_2$ will stand instead of F_1 and F_2 accordingly.

Let assign the following function:

$$\Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\cos\beta_1 + x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\sin\beta_1. \tag{16}$$

It makes possible to look at this procedure from two positions. In first, the transformation of the system has occurred at presence of the tracing (control) force in the system which provides the consideration of interesting classes of the motion (15). In second, it makes possible to look at this like the procedure which allows to deflate the system. Really, the system (2)–(5), (6)–(11) as a result of that action generates an independent system of the sixth order of the following type:

$$\dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 - \omega_5 v\cos\alpha\sin\beta_2 +$$

$$+\omega_4 v\cos\alpha\cos\beta_2 - \sigma\dot\omega_5\sin\gamma - \sigma\dot\omega_4\cos\gamma = 0, \tag{17}$$

$$\dot\alpha v\cos\alpha\sin\beta_1 + \dot\beta_1 v\sin\alpha\cos\beta_1 + \omega_3 v\cos\alpha\sin\beta_2 -$$

$$-\omega_2 v\cos\alpha\cos\beta_2 + \sigma\dot\omega_3\sin\gamma + \sigma\dot\omega_2\cos\gamma = 0, \tag{18}$$

$$(I_1 + I_3)\dot\omega_2 = -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\cos\gamma, \tag{19}$$

$$(I_1 + I_3)\dot\omega_3 = -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\sin\gamma, \tag{20}$$

$$(I_1 + I_3)\dot\omega_4 = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\cos\gamma, \tag{21}$$

$$(I_1 + I_3)\dot\omega_5 = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2\sin\gamma, \tag{22}$$

in which the parameters v, β_2 are added to the constant parameters specified above.

2.2 Two systems of the discourses on integrability

Remark 1 (on analytical first integrals). Obviously that the system (17)–(22) possesses two analytical first integrals which are expressed in terms of the finite combination of the elementary functions:

$$\omega_2 \sin\gamma - \omega_3 \cos\gamma = W_1' = \text{const}, \tag{23}$$

$$\omega_4 \sin\gamma - \omega_5 \cos\gamma = W_2' = \text{const}. \tag{24}$$

First of all this means that the system (17)–(22) can be reduced to the fourth order system on its own four-dimensional phase manifold.

Hereafter, it makes possible to develop by the following ways under the study of the system (17)–(22) (i.e. to accept the following systems of the discourses).

I. In first, it makes possible "not to notice" the existence in the system the first integrals of the forms (23), (24). Then conducting the series of the equivalent transformations it can possible try to reduce the investigated system (17)–(22) to the equivalent system in which the reduction to the systems of lower dimensionality will occur. Herewith, it is sufficient to get the quantity of the independent first integrals smaller then previous one on two units for the complete system integration, by virtue of (23), (24).

II. In second, it makes possible to use the first integrals (23), (24) expressing two interested phase variables from the list $\omega_2, \omega_3, \omega_4, \omega_5$. Herewith, we shall get just the fourth order system as the system which is the reduction of the system (17)–(22) to the certain four-dimensional phase manifold.

In the beginning we shall choose the system of the discourses **I**.

Really, the system (17)–(22) is equivalent to

$$\begin{gathered}
\dot\alpha v \cos\alpha - \omega_5 v \cos\alpha\cos\beta_1\sin\beta_2 + \omega_4 v\cos\alpha\cos\beta_1\cos\beta_2 + \\
+\omega_3 v\cos\alpha\sin\beta_1\sin\beta_2 - \omega_2 v\cos\alpha\sin\beta_1\cos\beta_2 - \\
-\sigma\dot\omega_5\sin\gamma\cos\beta_1 - \sigma\dot\omega_4\cos\gamma\cos\beta_1 + \sigma\dot\omega_3\sin\gamma\sin\beta_1 + \sigma\dot\omega_2\cos\gamma\sin\beta_1 = 0,
\end{gathered} \tag{25}$$

$$\begin{gathered}
\dot\beta_1 v\sin\alpha + \omega_3 v\cos\alpha\cos\beta_1\sin\beta_2 - \omega_2 v\cos\alpha\cos\beta_1\cos\beta_2 + \\
+\omega_5 v\cos\alpha\sin\beta_1\sin\beta_2 - \omega_4 v\cos\alpha\sin\beta_1\cos\beta_2 + \\
+\sigma\dot\omega_3\sin\gamma\cos\beta_1 + \sigma\dot\omega_2\cos\gamma\cos\beta_1 + \sigma\dot\omega_5\sin\gamma\sin\beta_1 + \sigma\dot\omega_4\cos\gamma\sin\beta_1 = 0,
\end{gathered} \tag{26}$$

$$\dot\omega_2 = -\frac{v^2}{I_1+I_3} x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)\cos\gamma, \tag{27}$$

$$\dot\omega_3 = -\frac{v^2}{I_1+I_3} x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)\sin\gamma, \tag{28}$$

$$\dot\omega_4 = \frac{v^2}{I_1+I_3} x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)\cos\gamma, \tag{29}$$

$$\dot\omega_5 = \frac{v^2}{I_1+I_3} x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)\sin\gamma, \tag{30}$$

Let introduce new quasivelocities in the system. We shall transform the values $\omega_2, \omega_3, \omega_4, \omega_5$ by means of the composition of the following rotations for this:

$$\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} = T_*(-\beta_1)\begin{pmatrix} \omega_3 \\ \omega_5 \end{pmatrix}, \quad \begin{pmatrix} z_3 \\ -z_4 \end{pmatrix} = T_*(-\beta_1)\begin{pmatrix} \omega_2 \\ \omega_4 \end{pmatrix}, \tag{31}$$

where

$$T_*(\beta_1) = \begin{pmatrix} \cos\beta_1 & -\sin\beta_1 \\ \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \tag{32}$$

and also

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = T_*(\beta_2) \begin{pmatrix} z_3 \\ z_1 \end{pmatrix}, \quad \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = T_*(-\beta_2) \begin{pmatrix} -z_4 \\ z_2 \end{pmatrix}. \tag{33}$$

Thus, the following relations are correct:

$$\begin{aligned} &z_1 = \omega_3 \cos\beta_1 + \omega_5 \sin\beta_1,\ z_2 = \omega_3 \sin\beta_1 - \omega_5 \cos\beta_1, \\ &z_3 = \omega_2 \cos\beta_1 + \omega_4 \sin\beta_1,\ z_4 = \omega_2 \sin\beta_1 - \omega_4 \cos\beta_1, \\ &w_1 = -z_1 \sin\beta_2 + z_3 \cos\beta_2,\ w_2 = z_3 \sin\beta_2 + z_1 \cos\beta_2, \\ &w_3 = z_2 \sin\beta_2 - z_4 \cos\beta_2,\ w_4 = z_4 \sin\beta_2 + z_2 \cos\beta_2. \end{aligned} \tag{34}$$

As is seen from (25)–(30), on the manifold

$$O_2 = \left\{ (\alpha, \beta_1, \omega_2, \omega_3, \omega_4, \omega_5) \in \mathbf{R}^6 :\ \alpha = \frac{\pi}{2}k,\ k \in \mathbf{Z} \right\} \tag{35}$$

it is impossible to resolve the system uniquely relatively to $\dot{\alpha}$, $\dot{\beta}_1$. Thus, the violation of the uniqueness theorem is happened on the manifold (35) formally. Moreover, in first, the indefiniteness is happened for even or odd k by the reason of degeneration of the coordinates $(v, \alpha, \beta_1, \beta_2)$ which are parameterized the three-dimensional sphere (but are not the classical spherical coordinates), and, in second, it is happened the evident violation of the uniquiness theorem for odd k because of the first equation of (25) degenerates for this case.

It follows that the system (25)–(30) outside of and only outside of the manifold (35) is equivalent to the system

$$\dot{\alpha} = -w_3 + \frac{\sigma v}{I_1 + I_3} \frac{s(\alpha)}{\cos\alpha} \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right), \tag{36}$$

$$\dot{w}_4 = -\frac{v^2}{I_1 + I_3} s(\alpha) \sin(\beta_2 + \gamma) \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) + $$
$$+ w_2 \left[w_1 \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1 + I_3} \frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \right], \tag{37}$$

$$\dot{w}_3 = \frac{v^2}{I_1 + I_3} s(\alpha) \cos(\beta_2 + \gamma) \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) - $$
$$- w_1 \left[w_1 \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1 + I_3} \frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \right], \tag{38}$$

$$\dot{w}_2 = \frac{v^2}{I_1 + I_3} s(\alpha) \sin(\beta_2 + \gamma) \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) - $$
$$- w_4 \left[w_1 \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1 + I_3} \frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) \right], \tag{39}$$

$$\dot{w}_1 = \frac{v^2}{I_1 + I_3} s(\alpha)\cos(\beta_2 + \gamma) \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) +$$

$$+w_3 \left[w_1 \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1 + I_3} \frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right], \tag{40}$$

$$\dot{\beta}_1 = w_1 \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_1 + I_3} \frac{s(\alpha)}{\sin\alpha} \cdot \Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right), \tag{41}$$

where

$$\Pi_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = -x_{4N}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\cos\beta_1 + x_{3N}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\sin\beta_1, \tag{42}$$

and the function $\Lambda_{v,\beta_2}\big(\alpha, \beta_1, \Omega/v\big)$ is represented in the form (16).

2.3 Case of the dependence of the moment of the nonconservative forces on the angular velocity

2.3.1 Introduction on the dependence on the angular velocity

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ are the coordinates of the point N of the action of the non-conservative force (of a medium interaction) to two-dimensional disk, $Q = (Q_1, Q_2, Q_3, Q_4)$ are the components not depending on the angular velocity tensor. We shall introduce the dependence of the functions $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the angular velocity tensor by the linear form only since given introduction itself is not obvious a priori.

And so, let accept the following dependence: $x = Q + R$, where $R = (R_1, R_2, R_3, R_4)$ is the vector-function containing the components of angular velocity tensor. Herewith, the dependence of the function R on the angular velocity tensor is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = -\frac{1}{v}\begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}. \tag{43}$$

Here (h_1, h_2, h_3, h_4) are the certain positive parameters.

And now, with the reference to our problem, since $x_{1N} \equiv x_{2N} \equiv 0$, then

$$x_{3N} = Q_3 - \frac{h_1}{v}(\omega_4 - \omega_5),\ x_{4N} = Q_4 - \frac{h_1}{v}(\omega_3 - \omega_2). \tag{44}$$

2.3.2 Reduced system

Similarly to the choice of the Chaplygin analytical functions

$$Q_3 = A\sin\alpha\cos\beta_1,\ Q_4 = A\sin\alpha\sin\beta_1,\ A > 0, \tag{45}$$

we shall accept the dynamic functions s, x_{3N} and x_{4N} as the following form:

$$s(\alpha) = B\cos\alpha,\ B > 0,\ h = h_1 > 0,\ v \neq 0,\ h = h_2 > 0,$$

$$x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\cos\beta_1 - \frac{h}{v}(\omega_4 - \omega_5), \tag{46}$$

$$x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\sin\beta_1 - \frac{h}{v}(\omega_3 - \omega_2),$$

which convinces us that the additional dependence of the damping moment of the non-conservative forces (and the dispersing one in some domains of the phase space) is also present in considered system (i.e. the dependence of the moment on the angular velocity tensor is present). Moreover, $h_1 = h_2,\ h_3 = h_4$ by virtue of the dynamical symmetry (12) of the body. Later on, let accept the system of discourses **I** which takes into account and the system of discourses **II** (see above).

We shall arouse to introduce the following variables in this section:

$$\begin{aligned} &u_1 = \omega_2 - \omega_3,\ u_2 = \omega_4 - \omega_5, \\ &u_3 = \omega_2 \cos\beta_2 - \omega_3 \sin\beta_2,\ u_4 = \omega_4 \cos\beta_2 - \omega_5 \sin\beta_2. \end{aligned} \tag{47}$$

Really, the assigned coordinates are defined correctly for $\cos\beta_2 \neq \sin\beta_2$, and Jacobian of the mapping is equal to $-(\cos\beta_2 - \sin\beta_2)^{-2}$, herewith, the inverse transformation is assigned as follows:

$$\begin{aligned} &\omega_2 = \frac{u_3 - u_1 \sin\beta_2}{\cos\beta_2 - \sin\beta_2},\ \omega_3 = \frac{u_3 - u_1 \cos\beta_2}{\cos\beta_2 - \sin\beta_2}, \\ &\omega_4 = \frac{u_4 - u_2 \sin\beta_2}{\cos\beta_2 - \sin\beta_2},\ \omega_5 = \frac{u_4 - u_2 \cos\beta_2}{\cos\beta_2 - \sin\beta_2}, \end{aligned} \tag{48}$$

and the particular case $\cos\beta_2 = \sin\beta_2$, which simplifies the dynamic equations can be considered separately.

Then the equations (25)–(30) under the condition (46) outside of and only outside of the manifold

$$O_3 = \left\{ (\alpha, \beta_1, \omega_2, \omega_3, \omega_4, \omega_5) \in \mathbf{R}^6 :\ \alpha = \frac{\pi}{2} + \pi k,\ k \in \mathbf{Z} \right\} \tag{49}$$

transform to the following equations:

$$\dot\alpha - u_3 \sin\beta_1 + u_4 \cos\beta_1 - \sigma n_0^2 v \sin\alpha + \sigma H_1' [-u_1 \sin\beta_1 + u_2 \cos\beta_1] = 0, \tag{50}$$

$$\dot\beta_1 \sin\alpha - \cos\alpha[u_3 \cos\beta_1 + u_4 \sin\beta_1] - \sigma H_1' \cos\alpha[u_1 \cos\beta_1 + u_2 \sin\beta_1] = 0, \tag{51}$$

$$\dot u_1 = -n_0^2 v^2 r_1 \sin\alpha \cos\alpha \sin\beta_1 - \frac{Bvh}{I_1 + I_3} r_1 u_1 \cos\alpha, \tag{52}$$

$$\dot u_2 = n_0^2 v^2 r_1 \sin\alpha \cos\alpha \cos\beta_1 - \frac{Bvh}{I_1 + I_3} r_1 u_2 \cos\alpha, \tag{53}$$

$$\dot u_3 = -n_0^2 v^2 \sin\alpha \cos\alpha \sin\beta_1 \cos(\gamma + \beta_2) - \frac{Bvh}{I_1 + I_3} u_1 \cos\alpha \cos(\gamma + \beta_2), \tag{54}$$

$$\dot u_4 = n_0^2 v^2 \sin\alpha \cos\alpha \cos\beta_1 \cos(\gamma + \beta_2) - \frac{Bvh}{I_1 + I_3} u_2 \cos\alpha \cos(\gamma + \beta_2), \tag{55}$$

where $r_1 = \cos\gamma - \sin\gamma \neq 0,\ n_0^2 = AB/(I_1 + I_3),\ H_1' = Bh/(I_1 + I_3)$.

Let introduce the following phase variables by the formulas:

$$\begin{aligned} &v_1 = -u_1 \sin\beta_1 + u_2 \cos\beta_1,\ v_2 = u_1 \cos\beta_1 + u_2 \sin\beta_1, \\ &v_3 = -u_3 \sin\beta_1 + u_4 \cos\beta_1,\ v_4 = u_3 \cos\beta_1 + u_4 \sin\beta_1. \end{aligned} \tag{56}$$

then outside of and only outside of the manifold

$$O_4 = \left\{ (\alpha, \beta_1, u_1, u_2, u_3, u_4) \in \mathbf{R}^6 :\ \beta_1 = \pi k,\ k \in \mathbf{Z} \right\} \tag{57}$$

the system (50)–(55) has the form

$$\dot{\alpha} = -v_3 - bH_1 v_1 + b\sin\alpha, \tag{58}$$

$$\dot{\beta}_1 = [v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{59}$$

$$\dot{v}_1 = n_0^2 v^2 r_1 \sin\alpha\cos\alpha - H_1' v r_1 v_1 \cos\alpha - v_2 \cdot [v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{60}$$

$$\dot{v}_2 = -H_1' v r_1 v_2 \cos\alpha + v_1 \cdot [v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{61}$$

$$\dot{v}_3 = n_0^2 v^2 \sin\alpha\cos\alpha\cos(\gamma+\beta_2) - H_1' v v_1 \cos\alpha\cos(\gamma+\beta_2) - v_4 \cdot [v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{62}$$

$$\dot{v}_4 = -H_1' v v_2 \cos\alpha\cos(\gamma+\beta_2) + v_3 \cdot [v_4 + bH_1 v_2]\frac{\cos\alpha}{\sin\alpha}, \tag{63}$$

where we introduce as before the dimensionless parameters as follows:

$$n_0^2 = \frac{AB}{I_1 + I_3},\ b = \sigma n_0,\ [b] = 1,\ H_1 = \frac{H_1'}{n_0} = \frac{Bh}{(I_1+I_3)n_0},\ [H_1] = 1. \tag{64}$$

Let also introduce one more auxiliary change of the part of the phase variables, as follows:

$$s_1 = v_3 + bH_1 v_1,\ s_2 = v_4 + bH_1 v_2. \tag{65}$$

Then the investigated system (58)–(63) after the introduction of dimensionless variables and differentiability $v_k \mapsto n_0 v v_k,\ k = 1, \ldots, 4,\ <\cdot>= n_0 v <'>$, will rewrite as the form

$$\alpha' = -s_1 + b\sin\alpha, \tag{66}$$

$$\beta_1' = s_2\frac{\cos\alpha}{\sin\alpha}, \tag{67}$$

$$s_1' = R_1 \sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} - R_1 H_1 v_1 \cos\alpha, \tag{68}$$

$$s_2' = s_1 s_2\frac{\cos\alpha}{\sin\alpha} - R_1 H_1 v_2 \cos\alpha, \tag{69}$$

$$v_1' = R_2 \sin\alpha\cos\alpha - s_2 v_2\frac{\cos\alpha}{\sin\alpha} - H_1 R_2 v_1 \cos\alpha, \tag{70}$$

$$v_2' = s_2 v_1\frac{\cos\alpha}{\sin\alpha} - H_1 R_2 v_2 \cos\alpha, \tag{71}$$

where $R_1 = bH_1(\cos\gamma - \sin\gamma) + \cos(\gamma+\beta_2),\ R_2 = r_1 = \cos\gamma - \sin\gamma$.

Obviously, that for $H_1 = 0$ formally the independent fourth order subsystem (66)–(69) stands out in the system (66)–(71) on the tangent stratification $T\mathbf{S}^2$ to two-dimensional sphere $\mathbf{S}^2\{0 < \alpha < \pi,\ 0 \le \beta_1 < 2\pi\}$, in which, in turn, it can be stand out the independent third order subsystem (66), (68), (69) on its own three-dimensional phase manifold.

And in the given case it is great for us that $H_1 \neq 0$. Therefore, we transform the having analytical first integrals (23), (24). We have the evident type of its in the different variables:

$$\frac{u_3 - u_1\sin\beta_2}{\cos\beta - 2 - \sin\beta_2}\sin\gamma - \frac{u_3 - u_1\cos\beta_2}{\cos\beta - 2 - \sin\beta_2}\cos\gamma = W_1' = \text{const}, \tag{72}$$

$$\frac{u_4 - u_2\sin\beta_2}{\cos\beta - 2 - \sin\beta_2}\sin\gamma - \frac{u_4 - u_2\cos\beta_2}{\cos\beta - 2 - \sin\beta_2}\cos\gamma = W_2' = \text{const}. \tag{73}$$

If we consider the case (15) (i.e., in particular, when the value β_2 is the identical constant along the phase trajectories), then the following analytical functions are constant on the phase trajectories of the considered system:

$$u_3(\sin\gamma - \cos\gamma) + u_1\cos(\gamma+\beta_2) = W_1^0 = \text{const}, \tag{74}$$

$$u_4(\sin\gamma - \cos\gamma) + u_2\cos(\gamma+\beta_2) = W_2^0 = \text{const}. \tag{75}$$

In another variables the latter two invariant relations have the forms

$$R_1 v_2\cos\beta_1 - R_1 v_1\sin\beta_1 + R_2[s_1\sin\beta_1 - s_2\cos\beta_1] = W_1^0 = \text{const}, \tag{76}$$

$$R_1 v_2\sin\beta_1 + R_1 v_1\cos\beta_1 - R_2[s_1\cos\beta_1 + s_2\sin\beta_1] = W_2^0 = \text{const}, \tag{77}$$

where $R_1 = \cos(\gamma+\beta_2) + bH_1(\cos\gamma - \sin\gamma)$, $R_2 = \cos\gamma - \sin\gamma$ as above.

Later on, let express from the relations (76), (77) the values v_1, v_2. We have:

$$v_2 R_1 = R_2 s_2 + \psi_1(\beta_1, W_1^0, W_2^0),\ v_1 R_1 = R_2 s_1 + \psi_2(\beta_1, W_1^0, W_2^0), \tag{78}$$

where

$$\begin{aligned}\psi_1(\beta_1, W_1^0, W_2^0) &= W_1^0\cos\beta_1 + W_2^0\sin\beta_1,\\ \psi_2(\beta_1, W_1^0, W_2^0) &= W_2^0\cos\beta_1 - W_1^0\sin\beta_1.\end{aligned} \tag{79}$$

Then the system (66)–(69) has the form of the independent fourth order system:

$$\alpha' = -s_1 + b\sin\alpha, \tag{80}$$

$$s_1' = R_1\sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} - R_2H_1s_1\cos\alpha - H_1\psi_2(\beta_1, W_1^0, W_2^0)\cos\alpha, \tag{81}$$

$$s_2' = s_1s_2\frac{\cos\alpha}{\sin\alpha} - R_2H_1s_2\cos\alpha - H_1\psi_1(\beta_1, W_1^0, W_2^0)\cos\alpha, \tag{82}$$

$$\beta_1' = s_2\frac{\cos\alpha}{\sin\alpha}. \tag{83}$$

The system (80)–(83) can be considered as the system (66)–(69) which is reduced to the levels (W_1^0, W_2^0) of the analytical first integrals (76), (77).

Obviously, that $\psi_1(\beta_1, 0, 0) \equiv \psi_2(\beta_1, 0, 0) \equiv 0$. Therefore, we shall consider the system (80)–(83) on the zero levels of the analytical first integrals (76), (77):

$$W_1^0 = W_2^0 = 0, \tag{84}$$

which has the form

$$\alpha' = -s_1 + b\sin\alpha, \tag{85}$$

$$s_1' = R_1\sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} - R_2H_1s_1\cos\alpha, \tag{86}$$

$$s_2' = s_1s_2\frac{\cos\alpha}{\sin\alpha} - R_2H_1s_2\cos\alpha, \tag{87}$$

$$\beta_1' = s_2\frac{\cos\alpha}{\sin\alpha}. \tag{88}$$

The given system can be considered on the tangent stratification $T\mathbf{S}^2$ to two-dimensional sphere $\mathbf{S}^2\{0 < \alpha < \pi,\ 0 \le \beta_1 < 2\pi\}$, in which, in turn, it can be stand out the independent third order subsystem (85)–(87) on its own three-dimensional phase manifold.

And so, for the integration of the sixth order system at the beginning we used the system of discourses **I** (see above), when we did not yet take into account the existence of two independent analytical first integrals of the forms (23), (24). In consequence we have limited (reduced) the considered sixth order system on the levels (in consequence zero) of the assigned first integrals, i.e. the system of discourses **II** was used (see above).

Theorem 1. *The system (2)–(5), (6)–(11) under the conditions (15), (46), (14), (84) possesses nine invariant relations (the complete tuple), three of which are the transcendental functions from the complex analysis view of point. Herewith, all the relations express in terms of the finite combination of the elementary functions.*

And at proof of the theorem 1 the system of discourses **II** is used (see above) which implies the reduction of investigated system on (zero) levels of the analytical first integrals (23), (24). The latter fact takes into account in principal the complete tuple of the having first integrals.

2.3.3 Topological analogies

Let consider the following third order system of the equations:

$$
\begin{aligned}
&\ddot{\xi} + (b_* - H_1^*)\dot{\xi}\cos\xi + R_3\sin\xi\cos\xi - \dot{\eta_1}^2\frac{\sin\xi}{\cos\xi} + \\
&+ H_1^{**}[W_1^0\sin\eta_1 - W_2^0\cos\eta_1] = 0, \\
&\ddot{\eta_1} + (b_* - H_1^*)\dot{\eta_1}\cos\xi + \dot{\xi}\dot{\eta_1}\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ H_1^{**}[W_1^0\cos\eta_1 + W_2^0\sin\eta_1] = 0,\ b_* > 0,\ H_1^{**} > 0,
\end{aligned}
\tag{89}
$$

describing the fixed spherical pendulum which is placed in a flow of the filling medium under the presence of the dependence of the moment of the forces on the angular velocity, i.e. the mechanical system in the nonconservative field of the forces. Unlike previous activities [1, 5], the order of such system is equal to 4 (but not 3) since the phase variable η_1 is not the cyclic, that does not reduce to the stratification of the phase space and the deflation.

Its phase space is the tangent stratification

$$
T\mathbf{S}^2\{\dot{\xi}, \dot{\eta_1}, \xi, \eta_1\}
\tag{90}
$$

to two-dimensional sphere $\mathbf{S}^2\{\xi, \eta_1\}$, herewith, the equation of the big circles $\dot{\eta_1} \equiv 0$ integral manifolds for $W_1^0 = W_2^0 = 0$ only.

It is not difficult to make sure that the system (89) is equivalent to the dynamic system with the (zero mean) variable dissipation on the tangent stratification (90) to two-dimensional sphere. Moreover, the following theorem is equitable.

Theorem 2. *The system (2)–(5), (6)–(11) under the conditions (15), (46), (14) is equivalent to the dynamic system (89).*

Really, it is sufficient to accept $\alpha = \xi$, $\beta_1 = \eta_1$, $b = -b_*$, $H_1 = H_1^{**}$, $R_2H_1 = -H_1^*$, $R_1 - bR_2H_1 = R_3$.

3 Conclusion

In the previous studies of the author, the problems on the motion of the four-dimensional solid were already considered in a nonconservative force field in the presence of the following force. This study opens a new cycle of works on integration of a multidimensional solid in the nonconservative field because previously, as was already specified, we considered only such motions of a solid when the field of external forces was the potential.

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