Finite element investigation of the gravitational and rotational deformation of the Earth

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Abstract

In this paper we investigate the deformation of Earth due to self-gravitation and constant rotation, *i.e.* centrifugal accelerations. A first rough estimate of a rotating, self-gravitating sphere made of iron using linear elasticity and infinitesimal strains shows that the deformations due to gravity alone are about 14 percent. These are comparatively large deformations. Consequently, we apply the concept of finite deformations instead and solve the local balance equations for mass and linear momentum in the current configuration, using the Almansi strain tensor. The stress and strain measures are related by a constitutive law similar to St.Venant-Kirchhoff. In order to solve this highly nonlinear problem, finite element calculations are conducted by using the research tool FEniCS [7]. Results show that the purely gravitational displacements are about two magnitudes larger compared to the ones from centrifugal forces, which has an impact on the accuracy of the so-called flattening parameter. We treat this problem by a thorough investigation of the magnitude of all participating terms, which leads to a decoupling of the system of highly non-linear differential equations. We compare the results with previously conducted analytical work presented in [9].

1 State of work and guide to the paper

The problem we investigate in this article is not completely new and was initially discussed by the masters of the old days, namely Newton [5, 2], Sommerfeld and Klein [4]. However, nowadays computational power is steadily increasing and numerical calculations enable us to tackle the subject by using advanced numerical techniques. Our initial investigations will concern the prediction of the so-called flattening parameter, which describes Earth's ellipsoidal shape. Newton himself found an approximate analytical expression for the flattening of Earth [2]:

$$f_{\text{Newton}} = \frac{a-c}{c} = \frac{5}{4}\frac{\omega_0^2\, a^3}{G\, m} = 4.35 \cdot 10^{-3} \ , \tag{1}$$

a and c being the equatorial and polar radius, ω_0 the angular velocity, m the mass of Earth and $G = 6.673 \cdot 10^{-11} {}^{\text{m}^3}/_{\text{kg}\,\text{s}^2}$ the gravitational constant. We will try to recover Newton's result by performing a 3D finite element analysis of an rotating sphere with the elastic properties of iron. This is the natural choice, since it is nowadays commonly accepted that Earth's inner core is essentially made of that. Moreover, initially we will assume the deformation to be small. Consequently, we will apply geometrically linear elastic theory in combination with an isotropic Hooke's law. First results will reveal deformations of a size that definitely challenges the validity of a geometrically linear theory. Hence, we will eventually switch to finite deformations, while concentrating exclusively on self-gravity, which is the dominant force to begin with. By doing so we end up with a radially symmetric, highly non-linear problem. Again, we will apply an isotropic constitutive relation for the Cauchy stresses, *i.e.*, $\boldsymbol{\sigma}$. However, this time we will base the deformation on the Green-Almansi strain tensor, $\boldsymbol{e}$, and assume a homogeneous initial mass distribution with an average *current* mass density, $\rho_0 = 5515\,{}^{\mathrm{kg}}/_{\mathrm{m}^3}$. This seems to be a reasonable estimate since mass is a conserved quantity. The governing differential equation will then be recast into dimensionless form containing Poisson's ratio, ν, and an additional parameter, $\alpha_k = \frac{4\pi G \rho_0^2 r_0^2}{3k}$, which accounts for the influence of self-gravitation and effective stiffness of the Earth (r_0 refers to Earth's average current outer radius). Since we have no direct knowledge regarding the effective bulk modulus, k, of Earth, we will use the modulus of iron leading to $\alpha_k = 1.976$. Unfortunately the direct finite element solution of the nonlinear problem, based on the Newton-Raphson algorithm, will not be successful. However, by incrementally increasing α_k we will be able to handle α_k-values of up to $\alpha_k = 1.1$ leading to displacements of about 20 % of the current outer radius, r_0. Results are shown for both linear and non-linear modeling in Section 4 after a continuum theory based description of the mechanical problem in Sections 2 and 3.

2 Theoretical background

In this section we derive the equations governing the problem of a self-gravitating rotating Earth. We start with the global balance equations for mass and linear momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} \rho\,\mathrm{d}V = 0\ , \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} \rho\boldsymbol{v}\,\mathrm{d}V = \oint_{\partial V_t} \boldsymbol{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}A + \int_{V_t} \rho\boldsymbol{f}\,\mathrm{d}V \tag{2}$$

ρ denoting the current mass density, $\boldsymbol{v}$ the velocity, $\boldsymbol{\sigma}$ the Cauchy stress tensor, $\boldsymbol{n}$ the outward normal vector, and $\boldsymbol{f}$ the specific body force. It is convenient to transform the equations onto a frame co-moving at the center of Earth, *i.e.*, rotating at a constant angular velocity $\boldsymbol{\omega} = \omega_0 \boldsymbol{e}_z$. To this end, we perform a Euclidian transformation [8, Chap. 8], where dashed quantities always describe entities in the co-moving frame:

$$\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{b}' \quad \text{or} \quad x_i\,(t)\,\boldsymbol{e}_i = x_i'\,(t)\,\boldsymbol{e}_i'\,(t) - b_i'\,(t)\,\boldsymbol{e}_i'\,(t)\,. \tag{3}$$

In this and all the following relations we use Einstein's summation convention. Now we investigate this equation component-wise and calculate the time derivatives. The change of basis vectors is a pure rotation with the angular velocity $\boldsymbol{\omega}$ where the origins O and O' of the systems coincide (*cf.*, Fig. 1):

$$\boldsymbol{v} = \dot{x}_i\boldsymbol{e}_i = \dot{x}_i'\boldsymbol{e}_i' + x_i'\boldsymbol{\omega}'\times\boldsymbol{e}_i'\ , \quad \boldsymbol{a} = \ddot{x}_i'\boldsymbol{e}_i' + 2\dot{x}_i'\boldsymbol{\omega}'\times\boldsymbol{e}_i' + x_i'\boldsymbol{\omega}'\times(\boldsymbol{\omega}'\times\boldsymbol{e}_i')\,. \tag{4}$$

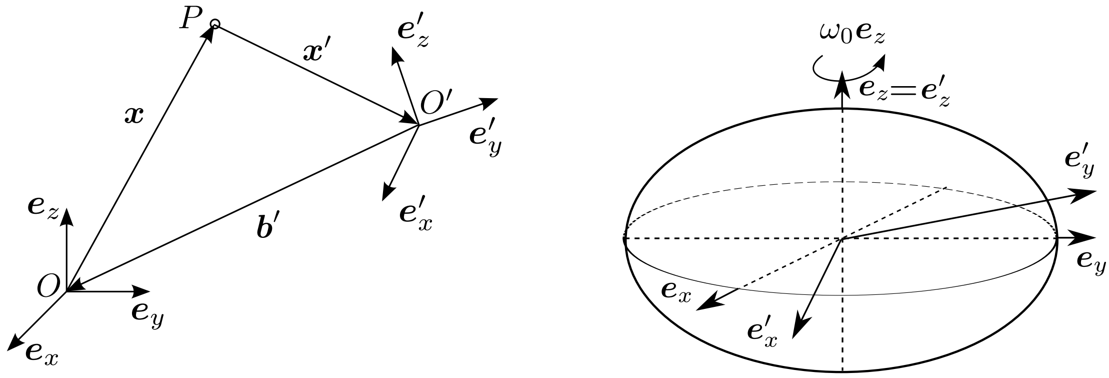

Figure 1: The co-moving basis vectors and quantities for the Euclidian transformation.

Since we use the co-moving system we may assume a stationary configuration, where Earth particles have no relative motion $\dot{x}'_i, \ddot{x}'_i = 0$ and therefore:

$$\boldsymbol{v} = \boldsymbol{\omega}' \times \boldsymbol{x}' \ , \quad \boldsymbol{a} = -\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \boldsymbol{x}') \, . \tag{5}$$

These results are necessary for obtaining the *local* balance equations. To this end we apply Reynold's transport theorem and assume the global balances to be valid for every sub-domain. Therefore the integrand itself has to be equal to zero:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \boldsymbol{v} = 0 \ , \quad \rho \boldsymbol{a} = \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}. \tag{6}$$

In context with Euclidian tensor properties of the occurring fields the following useful relations apply:

$$\rho = \rho' \ , \quad \nabla = \frac{\partial}{\partial x_i} \boldsymbol{e}_i = \frac{\partial}{\partial x'_i} \boldsymbol{e}'_i \ , \quad \boldsymbol{\sigma} = \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sigma'_{ij} \boldsymbol{e}'_i \otimes \boldsymbol{e}'_j \ , \quad \boldsymbol{f} = f_i \boldsymbol{e}_i = f'_i \boldsymbol{e}'_i. \tag{7}$$

By assuming a constant angular velocity in $\boldsymbol{e}'_3$ - direction, a further simplification results:

$$-\rho' \boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \boldsymbol{x}') = -\rho' \omega_0^2 \, (x'_1 \boldsymbol{e}'_1 + x'_2 \boldsymbol{e}'_2) = \nabla' \cdot \boldsymbol{\sigma}' + \rho' \boldsymbol{f}'. \tag{8}$$

The body forces $\boldsymbol{f}'$ result from self-gravity only. Since the gravitational force is conservative we may derive it from a scalar potential, U', obeying Poisson's equation:

$$\boldsymbol{f}' = -\nabla' U' \ , \quad \nabla' \cdot (\nabla' U') = 4\pi G \rho'. \tag{9}$$

For the sake of brevity we will omit dashes in what follows. We will see later that a formulation in the reference configuration is useful for deriving certain required relations for the unknown fields. The corresponding basic transformations read:

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}_0 \ , \quad \boldsymbol{n} \, \mathrm{d}A = J \boldsymbol{F}^{-T} \cdot \boldsymbol{n}_0 \, \mathrm{d}A_0 \ , \quad \mathrm{d}V = J \, \mathrm{d}V_0 \ , \tag{10}$$

with the deformation gradient:

$$\boldsymbol{F} = \frac{\partial z^i}{\partial Z^j} \boldsymbol{g}_i \otimes \boldsymbol{g}_0^j \ , \quad J = \det \boldsymbol{F} \ , \tag{11}$$

z^i and Z^j being the coordinates and $\boldsymbol{g}_i$ and $\boldsymbol{g}_0^j$ the (potentially) curvilinear base vectors in the current and referential configurations, respectively. With these identities we may write for the balance of mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0}\rho J\,\mathrm{d}V_0 = 0\ , \quad \frac{\mathrm{d}}{\mathrm{d}t}(\rho J) = 0\ , \quad \rho J = \rho_0 = \text{const.}\ , \tag{12}$$

and for the balance of linear momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0}\rho J\boldsymbol{v}\,\mathrm{d}V_0 = \oint_{\partial V_0} J\left(\boldsymbol{F}^{-T}\cdot\boldsymbol{n}_0\right)\cdot\boldsymbol{\sigma}\,\mathrm{d}A_0 \quad \text{or} \quad \rho_0\boldsymbol{a} = \nabla_{\boldsymbol{x}_0}\cdot\boldsymbol{P} + \rho_0\boldsymbol{f}. \tag{13}$$

Herein we have used the first Piola-Kirchhoff stress tensor $\boldsymbol{P} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}$.

3 Constitutive equations

In order to close the system of equations we have to specify constitutive relations, which connect the stress tensors $\boldsymbol{\sigma}$ or $\boldsymbol{P}$ with quantities of deformation. In the geometrically linear case we use well known Hooke's law:

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}\,(\boldsymbol{\varepsilon})\ \mathbf{1} + 2\mu\boldsymbol{\varepsilon} \quad \text{with} \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla_{\boldsymbol{x}}\otimes\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\otimes\boldsymbol{u})^T\right). \tag{14}$$

If we adopt the concept of finite deformations in the reference configuration we may use the St. Venant-Kirchhoff law, which relates the second Piola-Kirchhoff stress tensor, $\boldsymbol{S}$, and the Green-Lagrange strain tensor, $\boldsymbol{E}$:

$$\boldsymbol{S} = \lambda\,\mathrm{tr}\,(\boldsymbol{E})\ \mathbf{1} + 2\mu\,\boldsymbol{E}\ , \quad \boldsymbol{S} = J\ \boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\ , \quad \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\cdot\boldsymbol{F} - \mathbf{1}\right). \tag{15}$$

If we prefer to stay in the current configuration we may use a constitutive equation based on the Almansi strain tensor, $\boldsymbol{e}$:

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}\,(\boldsymbol{e})\ \mathbf{1} + 2\mu\,\boldsymbol{e}\ , \quad \boldsymbol{e} = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{B}^{-1}\right)\ , \quad \boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^T. \tag{16}$$

If infinitely small deformations are assumed, all three constitutive equations coincide, *cf.*, [1].

4 Results

In order to prepare the balance equations for a finite element analysis, we have to generate weak forms. Therefore we apply the procedure described in [6], *i.e.*, we multiply by a test function, $\delta\boldsymbol{u}$, integrate over the entire domain, and perform integration by parts:

$$\int_{V_t}\boldsymbol{\sigma}\cdot(\nabla_{\boldsymbol{x}}\otimes\delta\boldsymbol{u})\ \mathrm{d}V - \oint_{\partial V_t}\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\delta\boldsymbol{u}\,\mathrm{d}A = \int_{V_t}\rho\boldsymbol{f}\cdot\delta\boldsymbol{u}\,\mathrm{d}V + \int_{V_t}\rho\omega_0^2\,(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2)\cdot\delta\boldsymbol{u}\,\mathrm{d}V. \tag{17}$$

Analogously we derive for the potential, U, with a test function, δU:

$$-\int_{V_t} \nabla_{\boldsymbol{x}} U \cdot \nabla_{\boldsymbol{x}} \delta U \,\mathrm{d}V + \oint_{\partial V_t} \boldsymbol{n} \cdot \nabla_{\boldsymbol{x}} U \,\delta U \,\mathrm{d}A = \int_{V_t} 4\pi G \rho \,\delta U \,\mathrm{d}V. \tag{18}$$

For completion of the boundary value problem we require the following conditions to hold:

$$U|_{\boldsymbol{x}=\boldsymbol{0}} = 0 \;, \quad \boldsymbol{u}|_{\boldsymbol{x}=\boldsymbol{0}} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{n} \cdot \boldsymbol{\sigma}|_{\boldsymbol{x} \in \partial V_t} = \boldsymbol{0}. \tag{19}$$

In the last equation we neglect surface stresses since they are very small compared to the stresses in the interior. We simulated the geometrically linear case with linear continuous Galerkin elements (tetrahedra). Both Eqns. (17) and (18) were written in weak form and solved simultaneously. Additionally, we incorporated a spatially

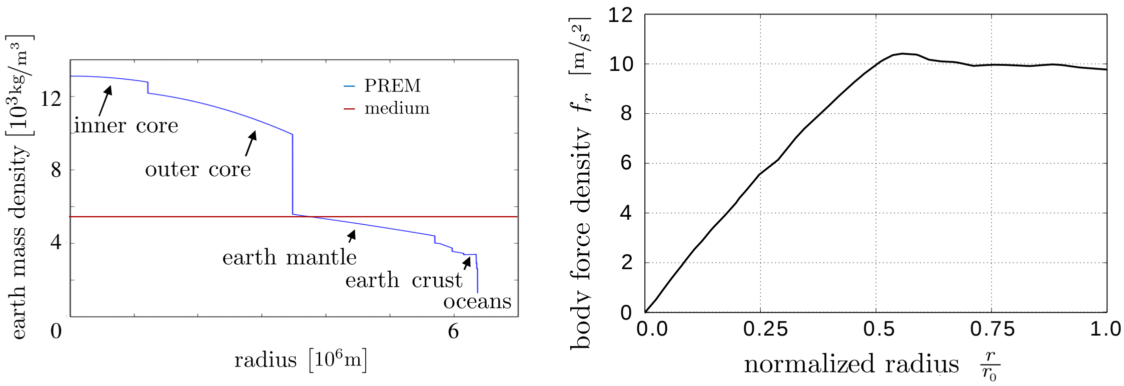

Figure 2: Mass density distribution according to PREM (left) and the corresponding gravitational acceleration (right) [3].

varying mass density, ρ, (see Fig. 2) of the PReliminary Earth Model (PREM, [3]) derived from a study of propagation velocities for seismic waves. The results in

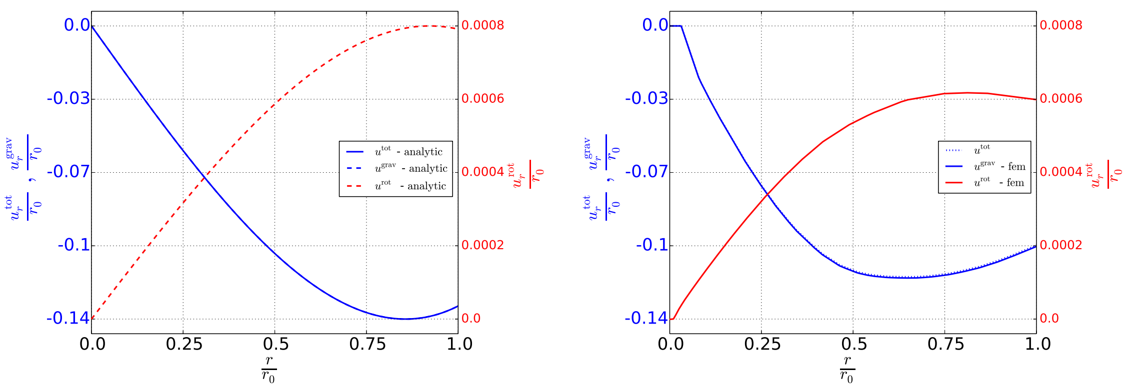

Figure 3: Analytic results from [9] based on a constant average mass density (left) and FE-results for normalized displacements along the equatorial plane from a 3D Cartesian simulation with the PREM mass density (right).

Fig. 3 show that for an earth-like celestial body deformations due to self-gravitation are beyond the validity of linear deformation theory. By comparing the curve for the total displacements, u_r^{tot}, *i.e.*, gravity plus centrifugal acceleration, and purely gravitational displacements, u_r^{grav}, there is almost no difference. By plotting the rotational displacements exclusively, we realize a difference of about two or three magnitudes. The flattening parameter of Eqn. (1) can now be calculated by taking the difference between displacements along the equatorial plane and poles:

$$f = \frac{u_r^{\text{rot}}\left(\vartheta = \pi/2\right) - u_r^{\text{rot}}\left(\vartheta = 0\right)}{r_0} = 1.1 \cdot 10^{-3}. \tag{20}$$

Consequently, from now on we concentrate on deformation by gravity alone and use non-linear theory. Now we have two possibilities: Either we transform all equations onto the reference configuration or onto the current configuration. The reference configuration has one major drawback, namely the unknown initial radius, R_0. A possible solution is to determine the radius iteratively by adjusting the radius in a stepwise manner, and always comparing the result to the real current outer radius. However, it is more convenient to transform all equations onto the current configuration and to use the real outer radius for the mesh. If we omit rotation we have a purely radially symmetric problem. Therefore, we switch to spherical coordinates and assume the deformation to be purely radial as well:

$$Z^1 = R\left(r\ ,\vartheta\ ,\varphi\right) = r - u_r\left(r\right)\ , \quad Z^2 = \Theta\left(r\ ,\vartheta\ ,\varphi\right) = \vartheta\ , \quad Z^3 = \Phi\left(r\ ,\vartheta\ ,\varphi\right) = \varphi, \tag{21}$$

with corresponding base vectors:

$$\boldsymbol{g}^i = \left\{\boldsymbol{e}_r\ , \frac{1}{r}\,\boldsymbol{e}_\vartheta\ , \frac{1}{r\sin\vartheta}\,\boldsymbol{e}_\varphi\right\} \quad \text{and} \quad \boldsymbol{g}_{0_k} = \left\{\boldsymbol{e}_R\ , R\boldsymbol{e}_\Theta\ , R\sin\Theta\boldsymbol{e}_\Phi\right\}. \tag{22}$$

This motion results in the following deformation measures:

$$\boldsymbol{F}^{-1} = \begin{bmatrix} \frac{\mathrm{d}R}{\mathrm{d}r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\boldsymbol{g}_{0_i}\otimes\boldsymbol{g}^k} = \begin{bmatrix} \left(1 - \frac{\mathrm{d}u_r}{\mathrm{d}r}\right) & 0 & 0 \\ 0 & \frac{R}{r} & 0 \\ 0 & 0 & \frac{R}{r} \end{bmatrix}_{\boldsymbol{e}_{0_i}\otimes\boldsymbol{e}^k}\ , \quad J = \left(1 - \frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^{-1}\left(\frac{r}{R}\right)^2. \tag{23}$$

With the help of the deformation gradient we may derive the only non-zero Green-Almansi strain components:

$$e_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r}\left(1 - \frac{1}{2}\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)\ , \quad e_{\vartheta\vartheta} = e_{\varphi\varphi} = \frac{u_r}{r}\left(1 - \frac{1}{2}\frac{u_r}{r}\right). \tag{24}$$

With Eqn. (16) we can now detail the Cauchy stress components. It is reasonable to normalize them by the bulk modulus, k:

$$t_{rr} = \frac{\sigma_{rr}}{k} = \frac{3\left(1-\nu\right)}{1+\nu}\frac{\mathrm{d}u_r}{\mathrm{d}r}\left(1 - \frac{1}{2}\frac{\mathrm{d}u_r}{\mathrm{d}r}\right) + \frac{2\nu}{1+\nu}\frac{u_r}{r}\left(1 - \frac{1}{2}\frac{u_r}{r}\right), \tag{25}$$

$$t_{\vartheta\vartheta} = t_{\varphi\varphi} = \frac{\sigma_{\vartheta\vartheta}}{k} = \frac{\sigma_{\varphi\varphi}}{k} = \frac{\nu}{1+\nu}\frac{\mathrm{d}u_r}{\mathrm{d}r}\left(1 - \frac{1}{2}\frac{\mathrm{d}u_r}{\mathrm{d}r}\right) + \frac{3}{1+\nu}\frac{u_r}{r}\left(1 - \frac{1}{2}\frac{u_r}{r}\right). \tag{26}$$

If the balance equation (13) is observed it turns out that only the radial component is different from zero. In addition to introducing normalized stresses, we define a dimensionless current radius, $x = {}^{r}\!/_{r_0}$, and a dimensionless displacement, $u = {}^{u_r}\!/_{r_0}$, by dividing both by the outer radius of the Earth, $r_0 = 6378\,\mathrm{km}$:

$$\frac{\mathrm{d}t_{rr}}{\mathrm{d}x} + \frac{1}{x}\left(2\,t_{rr} - t_{\vartheta\vartheta} - t_{\varphi\varphi}\right) = -\frac{\rho\, r_0}{k} f_r \;. \tag{27}$$

For a radial mass distribution Eqn. (9) reduces to:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}U}{\mathrm{d}r}\right) = 4\pi G\rho\left(r\right) \quad\Rightarrow\quad \frac{\mathrm{d}U}{\mathrm{d}r} = \frac{1}{r^2}\int_{\tilde{r}=0}^{\tilde{r}=r} 4\pi\rho\left(\tilde{r}\right)\tilde{r}^2\,\mathrm{d}\tilde{r}\;, \tag{28}$$

and therefore the gravitational accelerations are given by:

$$f_r = -\frac{Gm\left(r\right)}{r^2} = -\frac{Gm\left(R\right)}{r^2} = -\frac{G\,\frac{4}{3}\pi\rho_0 R^3}{r^2} = -\frac{4}{3}\pi G\rho_0\left(1-\frac{u_r}{r}\right)^3 r\;. \tag{29}$$

Herein we have used a homogeneous medium mass density, $\rho_0 = 5515\ {}^{\mathrm{kg}}\!/_{\mathrm{m}^3}$ (see Fig. 2). By inserting this result in Eqn. (27) and by using Eqn. $(23)_2$ we get:

$$\frac{\mathrm{d}t_{rr}}{\mathrm{d}x} + \frac{1}{x}\left(2\,t_{rr} - t_{\vartheta\vartheta} - t_{\varphi\varphi}\right) = \alpha_k\left(1-\frac{u}{x}\right)^5\left(1-\frac{\mathrm{d}u}{\mathrm{d}x}\right)x\;,\quad \alpha_k = \frac{4\,\pi\,G\,\rho_0^2\,r_0^2}{3\,k}. \tag{30}$$

Simulations were carried out for different parameters α_k, always keeping Poisson's ratio at $\nu = 0.3$ (Fig. 4, left) and $\nu = 0.38$ (Fig. 4, right), respectively. The results

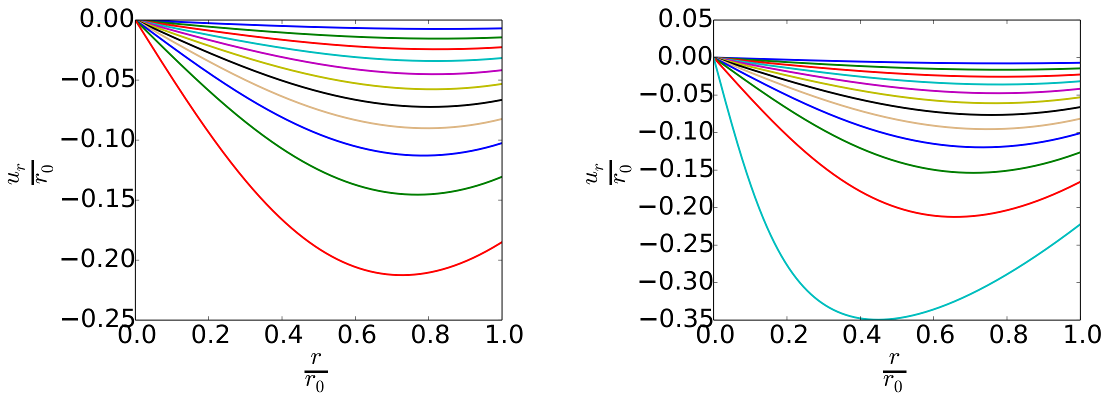

Figure 4: Nonlinear solution for $\nu = 0.3$, starting from $\alpha_k = 0.1$, upper blue line, to $\alpha_k = 1.1$, lower red line in steps of 0.1 (left), and for $\nu = 0.38$ and $\alpha_k = 0.1 \cdots 1.2$ (right).

(see Fig. 4) show once more very clearly that purely gravitational deformations of a massive, earth-like celestial body are beyond the validity of linear geometric theory. Therefore finite deformation theory was applied to the radially symmetric self-gravitational problem of a sphere and results are shown for different constellations of parameters. Unfortunately no convergence was achieved for the case when α_k turned to 1.976, which corresponds to $\alpha = 2.45$ when referring to the alternative mass-stiffness parameter $\alpha = \alpha_k \frac{2(1+\nu)}{3(1-\nu)}$ of the paper by Müller and Weiss [10] in the same proceedings. Their largest converging value of α is 1.76, obtained with the finite difference technique for $\nu = 0.38$. This corresponds to $\alpha_k = 1.18$, which is almost exactly the same value we were able to achieve (see Fig. 4 (right)) with the finite element algorithm. We tried to solve the differential equation by adaptively refining the mesh at regions were the displacement gradient is large, but to no avail: Convergence could not be achieved for larger values of α_k.

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