QUASICONFORMAL INSTANTONS

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The generalization of the standard chiral model for the case of a nontrivial topology (of the Riemann surface) of space–time is suggested. This model has an analog in the instanton solution. The Beltrami equation (quasiconformal mapping) arises naturally instead of Cauchy–Riemann equations (conformal mapping). The moduli space of quasiconformal instantons is connected with the Teichmüller space.

1. In recent times objects with nontrivial topological structure are considered in many problems, i.e. in gravitation [1], in the theory of dynamical systems with fractal dimensions (fractals) [2], in the Yang–Mills theory [3], in the string model [4]. This leads to a richer analytical structure than in the case of a trivial topological structure. In the chiral models, which are a two-dimensional laboratory for more realistic theories, moving away from the standard context of the theory of harmonic mappings [5] (the case of trivial topological structure) also leads to the more interesting analytical structure of the classical solutions of the corresponding Euler–Lagrange equations [6,7]. It should be noted that the harmonic mappings are exhausted by holomorphic mappings or are constructed from them [8] in the standard situation.

In this letter we shall consider a generalization of the standard chiral model for the case when the space–time manifolds (two-dimensional) have a nontrivial topological structure of the Riemann surface.

Our first goal is to generalize the instanton sector of the standard chiral model. Next (which will be considered separately), we are going to take into account the influence of the nontrivial topology on the quasiclassical calculations of $\int e^{-S} d\mu$ (S is an action functional) for the case of generalized instanton solutions considered in this letter. Such calculations for the standard model were performed in ref. [9].

The influence of the simplest nontrivial topology on the quantum effects in the four-dimensional case was considered by Isham [10].

The Riemann surface topology leads to consideration of quasiconformal mappings instead of more rigid conformal mappings. The analog of the conformal mapping (Cauchy–Riemann (CR) equations) in the model suggested is the quasiconformal mapping (Beltrami equation). The main properties of instantons of the standard chiral model are preserved in the generalized model. Exactly,

(a) any solution of the first order equation (Beltrami equation) is also a solution of the full equation of motion (Euler–Lagrange equation);

(b) this "ausiconformal instanton" is also parametrized by an arbitrary function;

(c) the lagrangians for these solutions are estimated through the jacobian.

This simple model shows the possibility of replacing a conformal mapping by a quasiconformal one while preserving the main properties and with enrichment of the analytical structure. In this way we naturally introduce the situation with topologically nontrivial space–time manifolds. In these generalized two-dimensional chiral models we can take the Riemann surface of the type (g,n,l), i.e. the surface of the genus g (with g handles) with n points removed and l disjoint nondegenerate continua, as a model of two-dimensional space–time manifold.

2. Let us consider the standard one-dimensional (complex Kähler chiral model [11] (the case of arbitrary dimension is considered analogously). Its lagrangian is

$$L = h(u,\bar{u})(u_z\bar{u}_z + u_{\bar{z}}\bar{u}_z), \tag{1}$$

where $z, \bar{z} \in \mathrm{M}$ (= C, C ∪ ∞ = S^2 or $\mathrm{M}_0 \in \mathrm{C}$) are coordinates of the two-dimensional (real) space–time manifold ($z = x_1 + \mathrm{i}x_2, \bar{z} = x_1 - \mathrm{i}x_2$), $u = u(z,\bar{z})$ is a chiral field. It is a local coordinate in some manifold G (complex one-dimensional hermitean manifold) with the metric $h = h(u,\bar{u})$.

If $\mathrm{M} = \mathrm{S}^2$ then we naturally introduce a topological charge with density

$$q = c^{-1} h(u,\bar{u})(u_z\bar{u}_{\bar{z}} - u_{\bar{z}}\bar{u}_z). \tag{2}$$

The Euler–Lagrange equation corresponding to the lagrangian (1) is

$$hu_{z\bar{z}} + (\partial h/\partial u)u_z u_{\bar{z}} = 0. \tag{3}$$

The instanton of the model is the solution of the CR equations

$$u_{\bar{z}} = 0 \quad [\text{or } v_z = 0]. \tag{4}$$

that is an arbitrary (anti)holomorphic function,

$$u = u(z) \quad [\text{or } v = v(\bar{z})]. \tag{5}$$

It is evident that eq. (3) is satisfied by (4) or (5). If we consider only the configurations with finite action then the chiral field is a meromorphic function that is characterized by a finite number of zeros and poles.

3. Now we consider constructions which allow us to give some natural generalization of the standard chiral model (1), (3) and its instanton sector (4), (5). This construction is motivated by the fact that the local instanton solution (5) [the solution of CR equations (4)] does not depend on the form of the metric h on the manifold G.

When we perform quasiclassical calculations, we have to substitute the explicit solution of the CR equations (instantons) in the expressions for the metric h and lagrangian L (1).

Let M, N be Riemann surfaces of the same type with local coordinates $z \in \mathrm{M}$ and $x \in \mathrm{N}$, G_h is a complex one-dimensional Kähler manifold with the local coordinate u and the metric $h = h(u,\bar{u})$. We suppose the following diagram of *local* mappings to be *commutative:*

$$\begin{array}{ccc} z \in \mathrm{M} & \xrightarrow{u} & u \in \mathrm{G}_h, \\ & {}_f\nwarrow \quad \nearrow_p & \\ & x \in \mathrm{N} & \end{array} \qquad u = p \circ f, \tag{6a}$$

where

$u: \mathrm{M} \to \mathrm{G}_h, \quad u = u(z,\bar{z}),$

$f: \mathrm{M} \to \mathrm{N}, \quad x = f(z,\bar{z}),$

$p: \mathrm{N} \to \mathrm{G}_h, \quad u = p(x,\bar{x}).$

Now we consider two cases.

(i) Let the mappings f and p be conformal, i.e. the CR equations are satisfied,

$$f_{\bar{z}} = 0 \quad \text{and} \; p_{\bar{x}} = 0.$$

It is clear, that u is also a conformal mapping. If the lagrangian, associated with this diagram is equal to (1), we have exactly the standard chiral model (1), (3) and its instanton sector (4), (5) with the extra pull back mapping p. It should be noted that any transfer to another set of local coordinates is a conformal (holomorphic) mapping.

(ii) Let mapping f be a *quasiconformal* mapping, i.e. the Beltrami equation [12] is satisfied (in local coordinates),

$$f_{\bar{z}} = \mu f_z,$$

where $\mu = \mu(z,\bar{z})$ is an arbitrary fixed function and p is a conformal mapping. As a consequence the mapping u also satisfies the Beltrami equation.

Let us consider the generalized lagrangian

$$L = h(u,\bar{u})[\alpha u_z\bar{u}_{\bar{z}} + \beta u_{\bar{z}}\bar{u}_z + \gamma(u_{\bar{z}} - \mu u_z)(\bar{u}_z - \bar{\mu}\bar{u}_{\bar{z}})] \tag{6b}$$

instead of (1). Here α, β, γ are constants, $\mu = \mu(z,\bar{z})$ is an arbitrary fixed function.

Property (a). It can be shown by a straightforward calculation that the Euler–Lagrange equation corresponding to lagrangian (6b),

$$hu_{z\bar{z}} + h\mu\bar{\mu}u_{z\bar{z}} - h\mu u_{zz} - h\bar{\mu}u_{\bar{z}\bar{z}} + (\partial h/\partial u)(u_z u_{\bar{z}} + u_z u_{\bar{z}}\mu\bar{\mu} - u_z^2\mu - u_{\bar{z}}^2\bar{\mu}) - h\mu_z u_z + h\mu_{\bar{z}}\mu u_z + h\mu\bar{\mu}_{\bar{z}} - \bar{\mu}_{\bar{z}} h u_{\bar{z}} = 0, \tag{7}$$

is satisfied on the Beltrami (first order) equation

$$u_{\bar{z}} = \mu u_z \tag{8}$$

if $\alpha = -\beta$ (elsewhere further $\alpha \equiv -\beta$).

Thus the quasiconformal mapping f [12] of arbitrary Riemann surfaces $M \xrightarrow{f} N$ is associated with the diagram (6a), lagrangian (6b) and eq. (8). Eq. (8) is the generalized "duality equation" [see also properties (b) and (c) below].

If $\mu = 0, \gamma = 2\alpha$, lagrangian (6b) and the "quasiconformal duality equation" (8) transform into (1), (4). It should be noted that lagrangian (6b) is conformal-invariant for general local holomorphic transformations of the coordinates,

$$z \to z' = \Psi(z), \quad \bar{z} \to \bar{z}' = \bar{\Psi}(\bar{z}),$$

if the new "field" $\mu = \mu(z, \bar{z})$ is transformed according to the formula

$$\mu(z, \bar{z}) = \mu(\Psi(z), \bar{\Psi}(\bar{z}))\, \bar{\Psi}_{\bar{z}}/\Psi_z,$$

i.e. μ is a coefficient function of the invariant $(-1, 1)$ form (Beltrami differential)

$$\mu = \mu(z, \bar{z})\, d\bar{z}/dz. \tag{9}$$

If we consider the metric of the form

$$h = h_1(u, \bar{u})\, h_2(z, \bar{z}),$$

then it is necessary for the preservation of this construction, that the function $h_z(z, \bar{z})$ satisfies also the Beltrami equation

$$\partial h_2/\partial \bar{z} = \mu\, \partial h_2 \partial z,$$

or $\alpha = 0$.

If we want the lagrangian to be conformal-invariant in this case it is necessary that

$$h_2(z, \bar{z}) = h_2(\Psi(z), \bar{\Psi}(\bar{z})).$$

4. Now we consider the other properties of the "duality equation" (8).

Property (b). All the solutions of eq. (8) are given by the formula

$$u = \phi[\hat{f}_0(z)], \tag{10}$$

where $u_0 = \hat{f}_0(z)$ is some particular solution and ϕ is an arbitrary holomorphic function. Thus the generalized instanton also depends on an arbitrary function. The asymmetric lagrangian (9b) has no anti-instanton solutions. However we can consider another lagrangian instead of (6b), which has only anti-instanton solutions. In this case we have the "antidual" equation

$$f_z = \nu_f \bar{f}_z,$$

where ν_f is the second complex deviation [12].

It is easy to construct the solution of eq. (8) in the case of the quasiconformal automorphism of the plane with the normalization condition

$$u(z) = z + O(|z|^{-1}) \quad (z \to \infty),$$

$\|\mu\|_{L_\infty} < 1$, according to the formula

$$u(z) = z - \pi^{-1} \iint\limits_{C} \frac{\rho(s)\, ds_1\, ds_2}{s - z} \equiv z + T\rho(z),$$

$s = s_1 + is_2$. Here $\rho(z)$ is the solution of the singular integral equation

$$\rho - \mu \Pi \rho = \mu,$$

where

$$\Pi \rho = -\pi^{-1} \iint\limits_{C} \frac{\rho(s)\, ds_1\, ds_2}{(s - z)^2} = \partial T\rho/\partial z.$$

We can solve this equation by the method of iteration,

$$\rho = \mu + \mu\Pi\mu + \mu\Pi(\mu\Pi\mu) + \dots .$$

We can see that here the analytical structure is more complicated compared with the standard instanton which is a meromorphic function.

It should be noted that if the function $\mu(z, \bar{z})$ defined in the disc $|z| < 1$ satisfies the inequality $|\mu(z, \bar{z})| \leqslant k < 1$ and depends analytically on complex parameters $s_1, \dots\dots, s_r$ then there exists a homeomorphic solution $u(z, \bar{z})$ of eq. (8) in some disc $|z| < \epsilon$ (for sufficiently small ϵ), and this solution is a holomorphic function of the parameters $s_1, \dots, s_r$.

Property (c). On the duality equation we have

$$L(u_{\bar{z}} = \mu u_z) = \alpha h(u, \bar{u})(u_z \bar{u}_{\bar{z}} - u_{\bar{z}} \bar{u}_z)$$

$$= \alpha h(u, \bar{u})\; J\left(\frac{u \,|\, \bar{u}}{z \,|\, \bar{z}}\right) = \alpha h u_z \bar{u}_{\bar{z}}(1 - \mu\bar{\mu})$$

($|\mu| < 1$ in the theory of quasiconformal mappings). On the CR equation [holomorphic functions do not satisfy the full equation of motion (7)], we have

$$L(u_{\bar{z}} = 0) = h u_z \bar{u}_{\bar{z}}(\alpha + \gamma\mu\bar{\mu}),$$

$$L(u_z = 0) = h u_{\bar{z}} \bar{u}_z(\gamma - \alpha).$$

Lagrangian (6) is always positive ($\beta \equiv -\alpha$) for mappings preserving the orientation ($|u_{\bar{z}}| < |u_z|$).

According to the properties (a)–(c), it is natural to call a solution of the Beltrami equation associated with lagrangian (6b) and diagram (6a) a *quasiconformal instanton.*

5. As usual, when the group of gauge transformations is present (the group of conformal transformations in our case), it is of great interest to consider the moduli space for quasiconformal instantons, i.e. the space of classes of conformal equivalence $[f]$ for the quasiconformal homeomorphism.

It is known [12], that this space coincides with the space of equivalence classes of the marked Riemann surfaces [S] [with fixed basis in $\pi_1(S)$] is the Teichmüller space $T_{g,n,l}$ for surfaces of the type (g, n, l). It is the cover space for the space of equivalence classes of the Riemann surfaces:

$$R_{g,n,l} = T_{g,n,l}/\Gamma_{g,n,l},$$

where $\Gamma_{g,n,l}$ is the modular group of the $T_{g,n,l}$ space.

The Teichmüller space is a complex analytic manifold, but $R_{g,n,l}$ is only a normal complex space. It has singularities which come from the fixed points of $\Gamma_{g,n,l}$ corresponding to Riemann surfaces admitting conformal self-mappings.

It should be noted that $T_{g,n,l}$ is a real manifold, its dimension is $m = 6g - 6 + 2n + 3l$. There are also natural global complex coordinates in $T_{g,n}$. Namely $T_{g,n}$ with its complex structure can be realized as a bounded domain D_m in the complex space C^{m+1}, $m = 3g - 3 + n$. The local complex coordinates are called moduli.

For practical aims (for instance in quasiclassical calculations) we need some explicit representation for the moduli. Here the variation formulae may be useful. Let $\tau = (\tau_1, ..., \tau_m)$ be the local complex coordinates in the neighbourhood of some point $[S_0]$ in $T_{g,n}$ and $\tau([S_0]) = 0$, then

$$\tau_j = \iint_{S_0} \mu(z)\varphi_j(z)\,dx\,dy + O(\|\mu\|^2_{L_\infty}),$$

$j = 1, ..., m$ where $\varphi_1, ..., \varphi_m$ is a basis in the space of holomorphic quadratic differentials (relative to some finite generated fuchsian group), μ is a complex characteristics of the quasiconformal homeomorphism f of the surface S_0 on the surface $S^\mu = f(S_0)$ with coordinates τ and the valuation is steady in some neighbourhood of the point $[S_0]$. This formula is correct for an arbitrary point $[S_0]$ in $T_{g,n}$ and arbitrary coordinate system τ.

It should be noted that the Beltrami equation was considered in connection with the chiral O(2, 1) σ-model and with the Ernst equation in gravitation [13].

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