REGIONS OF PHASE TRANSITIONS IN NONLINEAR-ELASTIC ISOTROPIC MATERIALS. 2. INCOMPRESSIBLE MATERIALS WITH A POTENTIAL DEPENDING ON ONE OF THE DEFORMATION TENSOR INVARIANTS

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A coexistence of two phases of a nonlinear-elastic incompressible material with a potential depending only on one of the deformation tensor invariants is considered on the basis of the general relations obtained in [1]. The equilibrium phase boundary is a deformation discontinuity surface where besides the kinematic (continuity of the displacement field) and the force (force continuity) conditions, the energy condition is held [2], (see also References to [1]), as an additional restriction on possible discontinuous solutions. These conditions are considered as a system of equations for determining the normal to the deformation discontinuity surface and the deformations in one phase depending on those in the other one. The deformations under which the system has a solution form a region of phase transitions (PT).

Inside the PT region, the solution has the form of a one-parameter family excluding cases where the one-parameter representation vanishes because of the kinematic restrictions, for example, as in the case of the plane deformation, discussed in [3]. Splitting the system of equations enables us to separate out an equation connecting the deformation tensor invariants on a discontinuity, to construct a one-parameter family of the normals to the discontinuity surface, and to evaluate the corresponding discontinuities of a constraint reaction. Rules of the "lever" and "energy equidistantness" are formulated.

The one-parameter representation of the solution vanishes on the boundary of the PT's region, which is a surface of one-dimensional transitions in shear. At the same time, the maximum and minimum elongations determine the normal to the interphase boundary while the boundary coincides with the corresponding shear plane.

Here we consider a structure of the PT region, namely, correlation with a region of nonellipticity, zero discontinuity surfaces, and critical points. The Tresloir fracture potential is dealt with as an example. It is noted that there exist fundamental potentials satisfying identically the equation connecting the deformation invariants on the discontinuity surface.

1. INITIAL RELATIONS

Due to the discontinuity of the displacement field, the deformation gradients $\mathbf{F}^{\pm}$ (where $F_{ij} = \partial x_i / \partial X_j$ and $\mathbf{X}$ and $\mathbf{x}(\mathbf{X})$ are the vectors of point position in the initial and actual configuration) are related by Hadamard's condition (1.4*)[1] on the phase boundary:

$$\mathbf{F}^{+} = (\mathbf{E} + \mathbf{an}) \cdot \mathbf{F}^{-} \tag{1.1}$$

The dyad $\mathbf{an}$ determines the deformation discontinuity where $\mathbf{a}$ and $\mathbf{n}$ are the vectors of the jump amplitude and normals to the deformation discontinuity surface in the actual configuration, $\mathbf{E}$ is a unit

[1] The asterisk denotes a reference to formula of the Part 1 [1].

tensor. The signs (±) denote the parameters at opposite sides from the boundary. It is follows from (1.1) that

$$\mathbf{B}_+ = \mathbf{B}_- + \mathbf{a}\mathbf{n}\cdot\mathbf{B}_- + \mathbf{B}_-\cdot\mathbf{n}\mathbf{a} + N_1\mathbf{a}\mathbf{a} \tag{1.2}$$

where $\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^T$ is Finger's deformation measure (Cauchy-Green left-hand deformation tensor), $N_k = \mathbf{n}\cdot\mathbf{B}_-^k\cdot\mathbf{n}$ ($k \neq 0$ is an integer) are orientation invariants (we do not write the superscript (-) on N_k: $N_k^- = N_k$), any two of the N_k ($k = k_1, k_2$) determine the $\mathbf{n}$ uniquely when $\mathbf{B}_-$ is specified. For an incompressible material, $N_1^+ = N_1$ due to (2.11') (compare with [3]).

Since for an incompressible material $\det\mathbf{F} = 1$, taking into account (1.1) we have

$$\mathbf{a}\cdot\mathbf{n} = 0 \tag{1.3}$$

while the relations

$$[I_1] = 2\mathbf{a}\cdot\mathbf{B}_-\cdot\mathbf{n} + N_1\mathbf{a}\cdot\mathbf{a}, \quad [I_2] = \mathbf{a}\cdot\mathbf{B}_-^{-1}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{B}_-^{-1}\cdot\mathbf{n} \tag{1.4}$$

are true for jumps of the first $I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2$ and the second $I_2 = \lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2$ (λ_1 are the principal extensions) invariants of $\mathbf{B}$ according to (1.2) and (2.10') (the square brackets denote a jump on the phase boundary, $[\varphi] = \varphi_+ - \varphi_-$).

Since for a nonlinear-elastic incompressible material we have [4]

$$\mathbf{T} = -p\mathbf{E} + \mathbf{G} \tag{1.5}$$

where $\mathbf{G} = \mathbf{G}(\mathbf{F})$, the force (Poisson's condition) and energy conditions of equilibrium on the phase boundary take the form [1]:

$$[p] = -\mathbf{n}\cdot[\mathbf{G}]\cdot\mathbf{n} \tag{1.6}$$

$$\mathbf{\Pi}\cdot[\mathbf{G}]\cdot\mathbf{n} = 0 \tag{1.7}$$

$$[W] = \mathbf{a}\cdot\mathbf{G}^{\pm}\cdot\mathbf{n} \tag{1.8}$$

Here, $\mathbf{T}$ is Cauchy's stress tensor, p is a constraint reaction, W is a potential, $\mathbf{\Pi} = \mathbf{E} - \boldsymbol{nn}$ is a projector on the discontinuity surface.

In view of (1.1) and (1.3), conditions (1.7) and (1.8) form a system of three equations in four unknowns specifying the vectors $\mathbf{n}$ and $\mathbf{a}\perp\mathbf{n}$ as functions of the deformations in one of the phases (further, the phase (-)). Consequently, the solution has the form of a one-parameter family. The discontinuities of I_1, I_2, and p are evaluated by (1.4) and (1.6). The deformation gradients $\mathbf{F}$ for which the system has a solution form a region of phase transitions (PT). For an isotropic material, we have

$$\mathbf{G} = \mu_1\mathbf{B} + \mu_{-1}\mathbf{B}^{-1} \tag{1.9}$$

where $\mu_1 = 2W_1$, $\mu_{-1} = -2W_2$, and W_α ($\alpha = 1,2$) are the derivatives of the potential $W(I_1, I_2)$ with respect to I_α.

Poisson's condition for the normal force component (1.6) and relations (1.5), (1.9) yield reaction jump dependence (3.14') on the deformation and orientation invariants

$$[p] = 2[W_1]N_1 - 2[W_2]N_{-1} - 2W_2^+[I_2] \tag{1.10}$$

Poisson's condition (1.7) for the tangential force component takes the form (3.15')

$$W_1^+N_1\mathbf{a} + W_2^+\mathbf{\Pi}\cdot\mathbf{B}_-^{-1}\cdot\mathbf{a} = -[W_1]\mathbf{t}_1 + [W_2]\mathbf{t}_{-1} \tag{1.11}$$

($\mathbf{t}_k = \mathbf{\Pi}\cdot\mathbf{B}_-^k\cdot\mathbf{n}$), while the energy condition has the form (4.3')

$$[W] = 2W_1^- \mathbf{a}\cdot\mathbf{B}_-\cdot\mathbf{n} - 2W_2^- \mathbf{a}\cdot\mathbf{B}_-^{-1}\cdot\mathbf{n} \tag{1.12}$$

Further, we consider an incompressible material with the potential

$$W = W(I) \quad (I = I_1 \quad or \quad I_2) \tag{1.13}$$

Then,

$$\mathbf{T} = -p\mathbf{E} \pm \mu\mathbf{B}^{\pm 1} \tag{1.14}$$

where $\mu = 2W'$ is a shear modulus, the prime denotes a derivative with respect to I, the upper and lower signs ($\pm$) correspond to the cases $I = I_1$ and $I = I_2$.

Equations (1.13) and (1.10)-(1.12) yield expressions for both the reaction jump

$$[p] = \begin{cases} 2[W']N_1, & if \quad I = I_1 \\ -2([W']N_{-1} + W_+'[I], & if \quad I = I_2 \end{cases} \tag{1.15}$$

and the amplitude

$$\mathbf{a} = \frac{\omega - 1}{N_1}\mathbf{\Pi}\cdot\mathbf{B}_-\cdot\mathbf{n} \quad (\omega = W_-'/W_+') \tag{1.16}$$

(compare with (3.15*); we assume that $W_+' \neq 0$) as well as for the energy condition

$$[W] = \pm\mu\mathbf{a}\cdot\mathbf{B}_-^{\pm 1}\cdot\mathbf{n} \tag{1.17}$$

Remark. For $W_+' = 0$ Eq.(1.11) holds at $W = W(I)$ only when either $[W'] = 0$ or $\mathbf{t}_{\pm 1} = 0$, i.e. either $W'(I_+) = W'(I_-) = 0$ or $\mathbf{n} = \mathbf{e}_i$. At the same time, we have $W(I_+) = W(I_-)$ in view of (1.12). Thus, a coexistence with the deformation for which $W_+' = 0$ is possible solely for the nonmonotone function $W(I)$, with both of the quantities I_- and I_+ determined by the potential properties. For $W_+' = W_-' = 0$ we have $\mathbf{T}_+ = \mathbf{T}_- = -p\mathbf{E}$, the normal is an indeterminate. For the second case, we have $\mathbf{T}_+ = -p_+\mathbf{E}$ and $[p] = \mp 2W_-'\lambda_{i-}^2$.

From (1.16), it follows that

$$N_1\mathbf{a}\cdot\mathbf{a} = -\mathbf{a}\cdot\mathbf{B}_-\cdot\mathbf{n}[W']/W_+', \quad \mathbf{a}\cdot\mathbf{B}_-^{-1}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{B}_-^{-1}\cdot\mathbf{n}[W']/W_+' \tag{1.18}$$

Inserting (1.18) into (1.4) we obtain

$$[I_1] = \frac{W_+' + W_-'}{W_+'}\mathbf{a}\cdot\mathbf{B}_-\cdot\mathbf{n}, \quad [I_2] = -\frac{W_+' + W_-'}{W_+'}\mathbf{a}\cdot\mathbf{B}_-^{-1}\cdot\mathbf{n} \tag{1.19}$$

Taking into account the definition of N_k, substitution of (1.16) into (1.19) results in the equations

$$[I] = (\omega^2 - 1)\begin{cases} (N_1^{-1}N_2 - N_1), & I = I_1 \\ (N_{-1} - N_1^{-1}), & I = I_2 \end{cases} \tag{1.20}$$

which we write in the form

$$L(N) = V(I_+, I_-), \quad V(I_+, I_-) \triangleq (\omega^2 - 1)^{-1}[I] \tag{1.21}$$

$$L(N) \triangleq \begin{cases} N_1^{-1} N_2 - N_1, & I = I_1 \\ N_{-1} - N_1^{-1}, & I = I_2 \end{cases} \tag{1.22}$$

Note that for $W'_+ = W'_- \neq 0$ we have $\mathbf{a} = 0$ and $[I] = 0$ due to (1.11) and (1.4).

We emphasize that the function $L(N)$ does not depend on the specific type of the potential (1.13) and is universal in this sense.

Theorem. The region of PT's in an incompressible material with a potential depending on only one of the invariants of the deformation tensor, is determined by the joint solution of the equation

$$[W] = \frac{\mu_- \mu_+}{\mu_- + \mu_+} [I] \tag{1.23}$$

connecting I_- with I_+ and the inequality

$$0 \le V(I_+, I_-) \le k^2 \tag{1.24}$$

$$k^2 = (\lambda_{max}^{\pm 1} - \lambda_{min}^{\pm 1})^2 \tag{1.25}$$

where λ_{max} and λ_{min} are the maximum and minimum principal extensions in one of the phases, further, referred to the phase (-).

Proof. Equation (1.23) follows from the energy condition (1.17) and relations (1.19).

The function $L(N)$, which is linear with respect to one of the N_k, achieves its maximum and minimum on the boundary of the domain D of the allowable values of the orientation invariants N_k. At the $(i-j)$th side of the triangle D according to (4.8*), (4.9*), and (1.22), we have

$$L = \lambda_{i-}^{\pm 2} + \lambda_{j-}^{\pm 2} - \lambda_{i-}^{\pm 2} \lambda_{j-}^{\pm 2} N_1^{\mp 1} - N_1^{\pm 1}, \quad 0 \le L \le L_{ij} \tag{1.26}$$

$$L_{ij} \triangleq L(N_1^*) = (\lambda_{i-}^{\pm 1} - \lambda_{j-}^{\pm 1})^2, \quad N_1^* = \lambda_{i-} \lambda_{j-}, \quad n_k = 0$$

The function L takes on the minimum (zero) value at the triangle vertices for $\mathbf{n} = \mathbf{e}_i$ and it takes on the maximum value $L_{13} = (\lambda_{3-}^{\pm 1} - \lambda_{1-}^{\pm 1})^2$ for

$$N_1 = \lambda_{1-} \lambda_{3-} = \lambda_{2-}^{-1}, \quad n_2 = 0 \quad (\lambda_1 \le \lambda_2 \le \lambda_3) \tag{1.27}$$

when the normal lies on the principal plane $\mathbf{B}_-$ corresponding to the maximum and minimum extensions. Consequently, Eq. (1.21) has a solution on D iff inequality (1.24) holds.

Note that in the case of (1.27) when inequality (1.24) holds, we have

$$n_1^2 = \lambda_3^- / (\lambda_1^- + \lambda_3^-), \quad n_3^2 = \lambda_1^- / (\lambda_1^- + \lambda_3^-) \tag{1.28}$$

When $\mu_\pm > 0$, the left-hand side of (1.24) involves a nonconvexity (downwards) of the potential $W(I)$ on the segment $[I_-, I_+]$, i.e., a smaller value of the invariant on the discontinuity surface corresponds to a larger value of a shear modulus.

When the solution of Eq. (1.23) has several branches, a region of phase transitions is constructed for each branch.

Since for $J = 1$ ($J = \lambda_1 \lambda_2 \lambda_3$), we have

$$I = \lambda_1^{\pm 2} + \lambda_2^{\pm 2} + \lambda_3^{\pm 2} \tag{1.29}$$

the region of phase transitions for $I = I_2$ can be built up by the region of phase transitions for $I = I_1$ (for the same dependence $W(I)$) using the inversion $\lambda_i - \lambda_i^{-1}$.

For the two-dimensional jumps, e.g., on the plane $(\mathbf{e}_1, \mathbf{e}_3)$, the quantities λ_{min} and λ_{max} in (1.25) are replaced with λ_1 and λ_3; $\lambda_2 = \lambda$ is a parameter of the two-dimensional deformation, not necessarily an intermediate value of the principal extension.

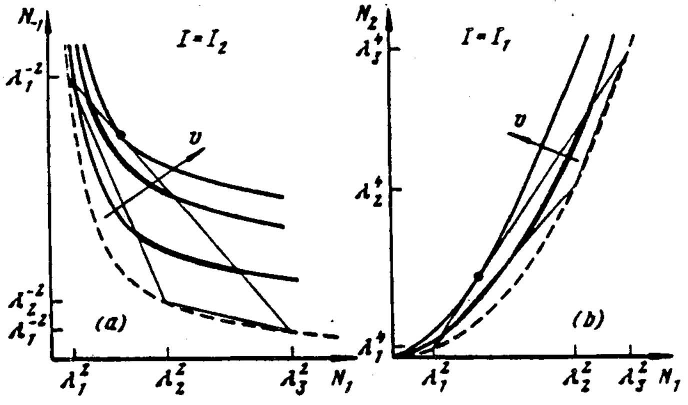

Fig. 1

Substituting the solution of Eq. (1.23) $I_+ = I_+(I_-)$ into (1.21), we have the equation

$$L(N) = \nu(I), \quad \nu(I) \overset{\Delta}{=} V(I_+(I), I) \tag{1.30}$$

specifying with view of D a one-parameter family of the normals. The constant I_+ given by (1.23), as well as the one-parameter families $[p]$ and $\mathbf{a}$ determined from (1.15) and (1.16), correspond to the family of normals. The form of the potential and solution of Eq. (1.23) determines the governing function $\nu(I)$.

At $I = I_1$, Eq.(1.30) specifies the parabola $N_2 = N_1^2 + \nu N_1$ in the plane (N_1, N_2) while, at $I = I_2$, it gives the hyperbola $N_{-1} = N_1^{-1} + \nu$ in the plane (N_1, N_{-1}), the curves being shifted skeleton curves [1]. When ν varies, the curves intersect three, two, one of the sides of the triangle D, touch externally upon sides 1-3 (when (1.27) holds), and, finally, go outside the D (Fig. 1).

Intersection of the $(i-j)$th side (plane jump) occurs necessarily at two points iff $0 < \nu(I) < L_{ij}$, and when $\nu(I) = L_{ij}$, the line (1.30) touches upon the $(i-j)$th sides. The existence of two different orientations of the discontinuity surface in the principal plane **B** (in contrast to, e.g., Hadamard's material [1]) is related to the behavior of the universal function L. That remains true for plane deformation of an arbitrary isotropic incompressible material (compare with [3]) since for the plane deformation the effective potential $W(I_1, I_2)$ [4] equivalent to the potential $W(I)$ can be always considered instead of it.

The energy condition has a very simple geometrical interpretation. Tangents to the curve $W(I)$ at the points I_- and I_+ are intersected at the point

$$I_0,\ W_0:\ W_0 - W_\pm = W_\pm'(I_0 - I_\pm) \quad (W_0 = W(I_0)) \tag{1.31}$$

From (1.23) and (1.31), the lever rule follows:

$$W_-'(I_- - I_0) = W_+'(I_0 - I_+) \tag{1.32}$$

(distantness of I_- and I_+ from I_0 is in inverse proportion to the shear moduli). This rule is equivalent to the rule of energy equidistantness, $W_+ - W_0 = W_0 - W_-$ (Fig. 2).

2. CONSTRUCTION OF THE REGION OF PHASE TRANSITIONS

The region of the phase transition is a two-dimensional region on the surface of incompressibility $\lambda_1\lambda_2\lambda_3 = 1$. Due to isotropy, it is adequate to construct it for

$$\lambda_1 \le \lambda_2 \le \lambda_3 \tag{2.1}$$

We choose the coordinates on which the construction of the phase transition region is simple. It follows from (1.25) and (1.29) that

$$k^2 = I - Y, \quad Y = \lambda_2^{\pm 2} + 2\lambda_2^{\mp 1} \ge 3 \tag{2.2}$$

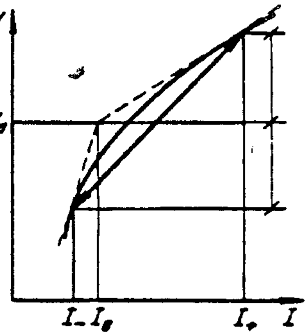

Fig. 2

Then inequality (1.24) takes the form

$$0 \le v(I) \le I - Y \tag{2.3}$$

Equality (2.3) nonlinear with respect to I is resolvable in the explicit form with respect to Y. This fact allows us to formulate

Statement 1. The region of phase transition in the plane (I, Y) is specified by the relations

$$3 \le Y \le r(I) \tag{2.4}$$

$$r(I) \le I \tag{2.5}$$

$$r(I_+, I) \triangleq I - V(I_+, I) = [\mu^2 I]/[\mu^2] = r(I, I_+) \tag{2.6}$$

$$r(I) \triangleq r(I_+(I), I) = I - v(I)$$

As follows from (2.6),

$$r(I_+) = r(I) \tag{2.7}$$

Let us determine the range of variables I and Y. It follows from (2.2) that

$$3 \le Y \le I \tag{2.8}$$

where $Y = 3$ for $\lambda_2 = 1$; $Y = 1$ for either $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in the three-dimiensional case or $\lambda_1 = \lambda_3 = \lambda^{-1/2}$ when $\lambda_2 = \lambda$ is a parameter of the plane deformation. While there is a one-to-one correspondence between points of the incompressibility surface and the coordinates I and λ_2 (namely, $\lambda_1 \lambda_3 = \lambda_2^{-1}$ and $\lambda_1^{\pm 2} + \lambda_3^{\pm 2} = I - \lambda_2^{\pm 2}$), two points (I, λ_2) for two values of $\lambda_2 < 1$ and $\lambda_2 > 1$, correspond to each point (I, Y) (Fig. 3, a). For the plane deformation, we have $\lambda_2 = \lambda$ then Y is specified. In the three-dimensional case, $\lambda_2 \in [\lambda_1, \lambda_3]$ is required; it results in contraction of the Y variable domain:

$$3 \le Y \le y(I) \tag{2.9}$$

where the function $y(I)$ conforms to the boundaries of sector (2.1) and is given parametrically (Fig. 3, b):

$$y = \lambda^{\pm 2} + 2\lambda^{\mp 1}, \quad I = 2\lambda^{\pm 2} + \lambda^{\mp 4} \tag{2.10}$$

where $\lambda = \lambda_1$ and $\lambda = \lambda_3$ for $\lambda_2 < 1$ and $\lambda_2 > 1$, respectively (e.g., if $\lambda_2 = \lambda_1$, then $\lambda_3 = \lambda_1^{-2}$ from what the second relation of (2.10) follows).

Consequently, to construct the phase transition region on coordinates, it is sufficient to determine a function $r(I)$ and to take into account the variable range of I and Y. Domains of the three-dimensional

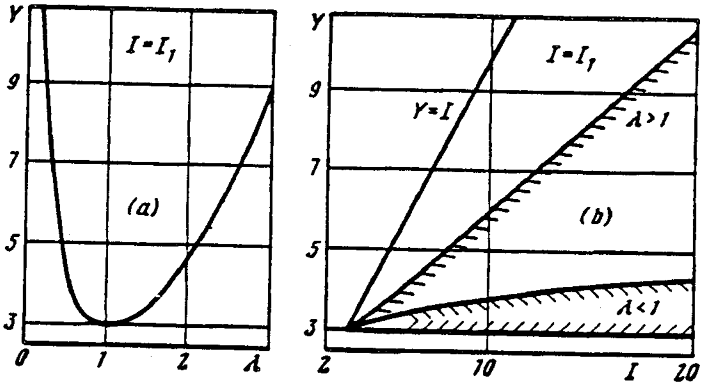

Fig. 3

discontinuities can be built up in accordance with the PT laws for the plane deformation by corresponding contraction of the variable range of I and Y.

Remark. Boundaries of the variable range of I and Y only bound the diagram $r(I)$. Sectors of different ordering of λ_i joint along the lines (2.10). The ray $Y = 3$, $I \geq 3$ is determined on the basis of Y and I. If region (2.4) lies inside (2.8), then condition (2.5) satisfies automatically.

3. PAIRS OF SUBREGIONS OF PHASE TRANSITIONS. MAXWELL'S RULE

Though the procedure proposed in section 2 solves completely a problem of construction of the PT region, it is of interest to consider the PT region on the basis of the dependence of the shear stress on the shear parameter [3, 5].

Since for deformations of the generalized simple shear in the plane (X_1, X_3) with unit vectors $\mathbf{i}_1$ and $\mathbf{i}_3$,

$$x_1 = \lambda^{-1/2}(X_1 + k_0 X_3), \quad x_3 = \lambda^{-1/2} X_3, \quad x_2 = \lambda X_2 \tag{3.1}$$

$$\mathbf{B} = \lambda^{-1}\left\{(1 + k_0^2)\mathbf{i}_1\mathbf{i}_1 + k_0(\mathbf{i}_1\mathbf{i}_3 + \mathbf{i}_3\mathbf{i}_1) + \mathbf{i}_3\mathbf{i}_3\right\} + \lambda^2\mathbf{i}_2\mathbf{i}_2$$

$$I_1 = \lambda^{-1}k_0^2 + \lambda^2 + 2\lambda^{-1} = \lambda_1^2 + \lambda_3^2 + \lambda^2$$

in view of the condition of incompressibility, we have

$$\lambda^{-1}k_0^2 = (\lambda_1 - \lambda_3)^2 = \lambda^2(\lambda_1^{-1} - \lambda_3^{-1}) \tag{3.2}$$

As a corollary from (1.25) and (3.1), the parameter $k = \lambda^{*1/2}k_0$ can be called a shear coefficient. The following expression corresponds to it:

$$\tau \triangleq W_k = \mu k \quad (W_k = \partial W(k^2 + Y)/\partial k) \tag{3.3}$$

and by (1.14) and (3.1) for the plane deformation $T_{13} = \lambda^{*1/2}\tau$; if $\lambda = 1$, then τ is the shear stress.

We assume that

$$I_A = \min(I_-, I_+), \quad I_B = \max(I_-, I_+) \tag{3.4}$$

Then according to (1.24) and (2.4), we have in the PT subregions $A = \{I_A, Y_A\}$ and $B = \{I_B, Y_B\}$:

$$\mu_A^2 > \mu_B^2 \tag{3.5}$$

$$Y_A \leq r, \quad Y_B \leq r \quad (r = r(I_A) = r(I_B)) \tag{3.6}$$

Inequalities (3.5) and (3.6) are equivalent to the condition

$$\mu_A^2(I_A - Y) = \mu_B^2(I_B - Y) \tag{3.7}$$

where $Y = Y_A$ or $Y = Y_B$.

Actually, according to (3.6), (2.6), and (3.5), we have $\mu_A^2 I_A - \mu_B^2 I_B > Y(\mu_A^2 - \mu_B^2)$, whence (3.7) follows.

If $Y = Y_A$, then, since by (2.8) $I_A > Y_A$, inequality (3.5) follows from (3.7), namely,

$$\mu_A^2/\mu_B^2 \geq (I_B - Y_A)/(I_A - Y_A) > 1 \tag{3.8}$$

For $Y = Y_B$, we have an analogous expression:

$$\mu_B^2/\mu_A^2 \leq (I_A - Y_B)/(I_B - Y_B) < 1 \tag{3.9}$$

In its turn, (3.6) follows from (3.7)-(3.9) and (2.6).

Condition (3.7) has a clear mechanical sense. According to (2.2) and (3.3), it means that

$$\tau^2(I_A, Y) \geq \tau^2(I_B, Y), \text{ if } I_A < I_B \tag{3.10}$$

or in variables k and Y,

$$\tau^2(k_A, Y) \geq \tau^2(k_B, Y), \text{ if } k_A^2 < k_B^2 \tag{3.11}$$

For the generalized plane deformation, we have $Y_A = Y_B = Y$; here τ is in proportion to the shear stress. For an arbitrary deformation state, τ is formally defined by equality (3.3). Now it is proved

Statement 2. A PT region is determined by the joint solution of inequality (3.11) and Eq. (1.23) in view of (2.9).

Note that we considered the intersection of the PT regions with lines $I = \text{const}$ in section 2 and with lines $Y = \text{const}$ in section 3.

Inequalities (1.24)$\leftrightarrow${(3.5), (3.6)}$\leftrightarrow$(3.7)$\leftrightarrow$(3.11) regardless of the energy condition (1.23) are different representations of necessary and sufficient conditions for the existence of the deformation discontinuity surface in the material (1.13) where both Poisson's and Hadamard's conditions hold. Inequality (3.11) implies a nonmonotonic character of the dependence $\tau(k)$ (when there is an increase section for $\tau(k) > 0$) and was obtained for the plane deformation $\lambda = 1$ of an incompressible material in [3, 5].

The vectors $\mathbf{n}$ and $\mathbf{a} \perp \mathbf{n}$ belonging on the PT region boundary $Y = r(I)$ lie in the principal plane of $\mathbf{B}_-$, in the plane (1-3). It is easy to show that in the basis ($\mathbf{n}$, $\boldsymbol{\tau}$, $\mathbf{e}_2$) (where $\mathbf{n}$ is determined by relations (1.28), $\boldsymbol{\tau} \perp \mathbf{n}$ is a unit vector in the principal plane, $\tau_1 = -n_3$ and $\tau_3 = n_1$), tensor $\mathbf{B}_-$ has the form (3.1):

$$\mathbf{B}_- = \lambda_{2-}^{-1}\left\{\mathbf{nn} + (1 + k_{0-}^2)\boldsymbol{\tau\tau} \pm k_{0-}(\mathbf{n}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{n})\right\} + \lambda_{2-}^2\mathbf{e}_2\mathbf{e}_2 \tag{3.12}$$

where $k_0 = \lambda_2^{1/2}(\lambda_3 - \lambda_1)$ and signs ($\pm$) correspond to cases when $n_1 n_3 \lesseqgtr 0$. Then due to (1.16) and (1.27), we have

$$\mathbf{a} = \pm(\omega - 1)k_{0-}\boldsymbol{\tau} \tag{3.13}$$

Substituting (3.12) and (3.13) into (1.2) we obtain

$$\mathbf{B}_+ = \lambda_{2-}^{-1}\left\{\mathbf{nn} + (1 + k_{0+}^2)\boldsymbol{\tau\tau} \pm k_{0+}(\mathbf{n}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{n})\right\} + \lambda_{2-}^2\mathbf{e}_2\mathbf{e}_2 \tag{3.14}$$

$$k_{0+} = \omega k_{0-} \tag{3.15}$$

whence it follows that tensors $\mathbf{B}_-$ and $\mathbf{B}_+$ at the PT region boundary differ only in value of the shear parameter, $\mathbf{a} = \pm[k_0]\boldsymbol{\tau}$, and the interphase boundary is a shear plane (such jumps are called normal in [3]). It allows us to built up the PT region boundary using Maxwell's rule for the one-dimensional transitions with respect to parameter k.

Actually, since $W_k^{\pm} = \mu_\pm k_\pm$, then the following equation can be derived from (3.15):

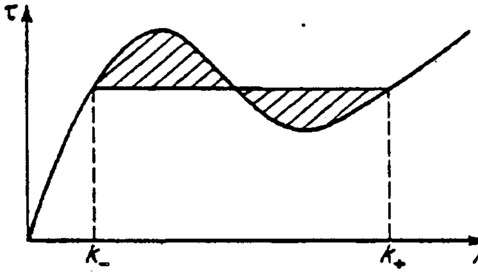

Fig. 4

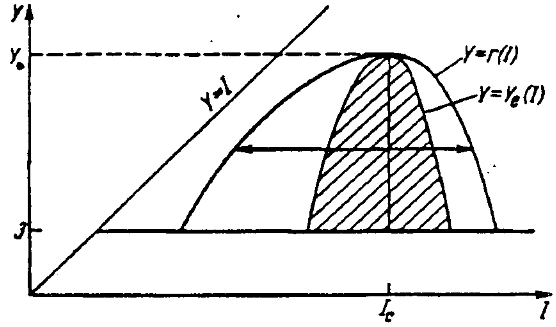

Fig. 5

$$W_k^- = W_k^+ \rightarrow \tau(k_-, Y) = \tau(k_+, Y) \tag{3.16}$$

that means the equality of the tangential forces on the discontinuity surface at the region boundary transitions.

Inserting (3.16) into (1.23) and taking into consideration that $[Y] = 0$ and $[I] = [k^2]$ for the transitions at the region boundary we obtain

$$[W] = W_k^- [k] \tag{3.17}$$

Equations (3.16) and (3.17) determine k_- and k_+ and can be written in the form of Maxwell's relations:

$$[W]/[k] = W_k^- = W_k^+$$

which mean the rule of equal areas on the graph of $\tau(k)$ (Fig.4). Having expressed k_- and k_+ against Y by Maxwell's relations we can further construct the region on the coordinates (I, Y). Such a procedure is alternative to the construction on the basis of inequalities (2.4).

4. ZERO-JUMP LINES AND ELLIPTICITY

The invariants I_- and I_+ appear in (1.23) symmetrically and any branch of the solution of Eq. (1.23) is symmetric about the straight line $I_+ = I_-$ in the plane (I_-, I_+). Intersection of the branch with the line $I_+ = I_-$ at $I_- = I_c$ corresponds to the zero-jump line on the surface of incompressibility. We can determine I_c from the condition of existence of a nontrivial zero solution of Eqs (1.23) and (1.21). Assuming $W(I)$ to be expanded into the Taylor series we obtain from (1.23) and (2.6):

$$\left(\frac{1}{3}\mu'' - \frac{\mu'^2}{\mu}\right)[I]^3 + \left(\frac{1}{6}\mu''' - \frac{\mu'\mu''}{\mu} + \frac{\mu'^3}{2\mu^2}\right)[I]^4 + \ldots = 0 \tag{4.1}$$

$$r(I, I_+) = I + \frac{\mu}{2\mu'} + \frac{3}{\mu'^2}\left(\mu'^2 - \frac{1}{3}\mu\mu''\right)[I] + \ldots \tag{4.2}$$

As follows from (4.1) and (4.2), the consideration of small jumps needs taking into account in the expansion of $W(I)$ the terms to the power no less than forth. The solution $I_+ = I_- = I_c$ exists when

$$\mu_c'^2 - \tfrac{1}{3}\mu_c\mu_c'' = 0 \tag{4.3}$$

$$Y \le r(I_c) = I_c + \mu_c/2\mu_c', \quad \mu_c/\mu_c' \le 0 \tag{4.4}$$

($\varphi_c = \varphi(I_c)$). For $\mu_c' \neq 0$ Eq. (4.3) is equivalent to

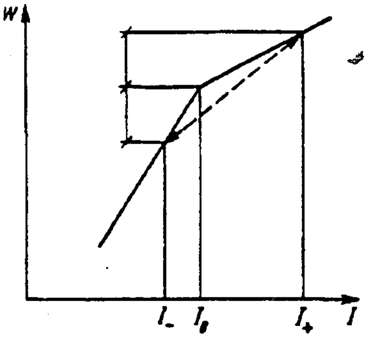

Fig. 6

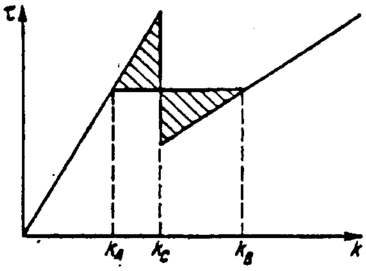

Fig. 7

$$(\mu/\mu')'_{I_c} = -2 \tag{4.5}$$

Relations (4.3) and (4.4) determine the internal boundary between the PT subregions.

Since according to (2.7) the value of r does not change at a PT, I_c is a point of either a local maximum or a local minimum of the continuous function $r(I)$ regardless of expensibility of $W(I)$ into the Taylor series.

Using the relations from [4] it is easy to show that local conditions of strict ellipticity (i.e., validity of Hadamard's inequality and positiveness of squares of plane wave velocities) on (I, Y) coordinates have the form

$$\mu > 0, \quad \mu + 2(I - Y)\mu' > 0 \tag{4.6}$$

or

$$\{\mu > 0, \quad \partial\tau^2(I, Y)/\partial I > 0\} \to \{\mu > 0, \quad \partial\tau(k, Y)/\partial k > 0\} \tag{4.7}$$

as in view of (3.3) and (2.2), we have

$$\partial\tau^2(I, Y)/\partial I = \mu(\mu + 2\mu'k^2), \quad \partial\tau(k, Y)/\partial k = \mu + 2\mu'k^2 \tag{4.8}$$

A condition of ellipticity in the form $d\tau(k)/dk > 0$ was formulated for the case of two-dimensional deformation in [5]. As follows from (3.10) and (4.6), the segment $\{[I_A, I_B], Y = \text{const} \in [3, r(I_A)]\}$ necessarily intersects the region of nonellipticity (instability).

The point (I_*, Y_*) of the nonellipticity region boundary

$$Y_e = I + \mu/(2\mu'), \quad \mu > 0 \tag{4.9}$$

where $Y_e'(I) = 0$, can be called critical. Differentiating (4.8) we obtain

$$(\mu/\mu')'_{I_*} = -2, \quad Y_* = Y_e(I_*) \tag{4.10}$$

At the critical point,

$$\partial\tau(k, Y_*)/\partial k = 0, \quad \partial^2\tau(k, Y_*)/\partial k^2 = 0 \tag{4.11}$$

The first condition follows directly from (4.8) and (4.9). Equations (4.8)-(4.9) yield the second one. Relations (4.11) are analogous to the dependence between pressure and volume at the critical point for "gas-liquid" phase transitions; here Y_* plays the role of "the critical temperature".

It follows from (4.5), (4.4), (4.9), and conditions (4.6), that: (*i*) the point of branching of the zero

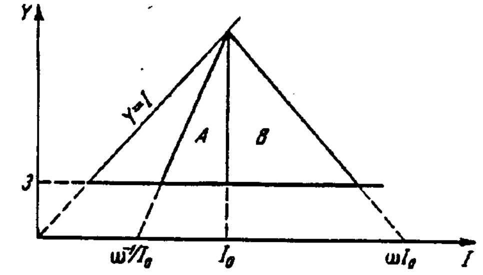

Fig. 8

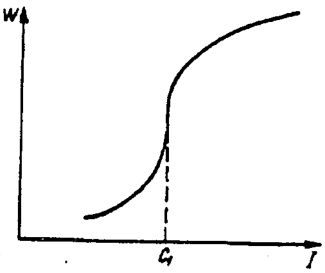

Fig. 9

solution of Eq. (1.23) coincides with the abscissa of the critical point:

$$I_* = I_c \tag{4.12}$$

(*ii*) the ordinate of the critical point is:

$$Y_* = r(Y_c) \tag{4.13}$$

(*iii*) the zero-jump line is the segment $\{I = I_c, 3 \le Y \le Y_*\}$ falling within the region of nonellipticity and limited by the critical point.

We emphasize that under the condition of ellipticity conservation at both sides of the discontinuity, discontinuities of deformation of infinitesimal amplitude are possible in materials with a continuous differentiable potential only in the vicinity of the critical point.

We consider, further, the characteristic features of the PT region when ellipticity can be conserved in the coexisting deformations. Differentiating Eq. (1.23) we can show that

$$dI_+/dI_- = -\zeta_-/\zeta_+, \quad \zeta = \mu + 2\mu' V(I_+, I) \tag{4.14}$$

As follows from (1.24) and (4.6), if Poisson's and Hadamard's conditions are valid on the discontinuity surface and the deformation $\mathbf{B}_-$ satisfies the ellipticity conditions then $\zeta_- > 0$ (if $\zeta_- < 0$, then conditions (4.6) are not satisfied in $\mathbf{B}_-$). A sign of ζ_+ and ellipticity in $\mathbf{B}_+$ are related in a similar way whence follows

Statement 3. If coexisting deformations satisfy the ellipticity conditions, then $dI_+/dI_- < 0$.

Since according to (2.6) we have $\partial r(I_+, I_-)/\partial I_\pm = \pm\mu_\pm\zeta_\pm/[\mu^2]$ then in view of (4.14) $dr/dI_- = -\zeta_-/[\mu]$. Consequently, if deformation $\mathbf{B}_-$ satisfies the ellipticity conditions on the discontinuity surface, then $dr/dI_- \lessgtr 0$ for $I_- \lessgtr I_+$, respectively, whence follows

Statement 4. If a deformation satisfies the ellipticity conditions at the PT boundary $Y = r(I)$, then

$$dY/dI_- \gtrless 0 \;\text{ if }\; I_- \lessgtr I_+ \tag{4.15}$$

A qualitative representation of the PT region boundaries where ellipticity can hold at the both sides of the discontinuity surface up to the zero-amplitude jumps is given in Fig. 5. The nonellipticity region is shaded.

5. EXAMPLE

We consider the Tresloir potential with a nonconvex angularity (Fig. 6)

$$W = \begin{cases} \mu_1 (I - 3)/2, & I \in [3, I_0] \\ (\mu_2 (I - I_0) + \mu_1 (I_0 - 3))/2, & I \in (I_0, \infty) \end{cases} \tag{5.1}$$

$$\mu_1 > \mu_2 > 0, \quad I_0 > 0$$

We determine the dependence $I_+(I_-)$ using the lever rule (1.32):

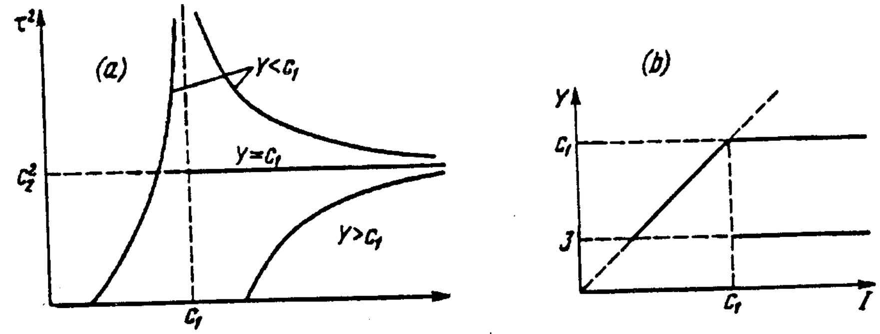

Fig. 10

$$I_+ = I_0 + \omega(I_0 - I_-) \quad (I_- < I_0, \quad \omega = \mu_1/\mu_2) \tag{5.2}$$

whence it follows that $I_c = I_0$ corresponds to the critical point.

In view of (1.30) and (5.2) a one-parameter family of the normals is specified by the equation $L(N) = (I_0 - I_-)/(\omega - 1)$.

According to (3.3) $\tau = \mu k$ where $\mu = \mu_1$ or $\mu = \mu_2$ when $k \in [0, k_c]$ or $k > k_c$, respectively; here $k_c^2 = I_c - Y$. Using Maxwell's rule of equal squares (Fig. 7) we observe that on the PT boundary $k_A = \omega^{-1/2} k_c$ and $k_B = \omega^{1/2} k_c (I_A \le I_B)$ whence, with view of (2.2), the following equations of the subregions A and B on coordinates under condition of ellipticity conservation at both sides of the discontinuity I and Y are derived (Fig. 8):

$$I_A = (1 - \omega^{-1})Y + \omega^{-1} I_c, \quad I_B = (1 - \omega)Y + \omega I_c$$

At a monotonous change of τ, a PT becomes first possible when $\tau = (\mu_1 \mu_2)^{1/2} (I_c - Y)^{1/2}$.

The instability region degenerates into the segment $\{I = I_c, Y \in [3, I_c]\}$ which is the internal boundary between the subregions A and B.

Note that the critical point $(I_c, Y = I_c)$ is unattainable for the three-dimensional jumps due to the region contraction of I-, Y-variable as result of λ_i ordering (see section 2).

6. ON FUNDAMENTAL POTENTIALS

Let us call the potentials $W(I)$, which for any I_- and I_+ satisfy Eq. (1.23) relating the deformation invariants at the both sides of the equilibrium discontinuity, the fundamental ones. We shall build them up taking Eq. (1.23) as a functional equation for $W(I)$. In this case, the PT region is defined by inequality (1.24) (or (3.10)).

Since (4.1) must hold for any I and for however small $[I]$, $W(I)$ necessarily satisfies Eq. (4.3) at any I. Obviously, the potential such that $W' = \text{const}$, (i.e., $W = AI + B$) satisfies conditions (1.23) and (4.3); but due to (1.20) it allows only zero jumps of I.

For $\mu \neq \text{const}$, integrating (4.5) we obtain

$$\mu^2 = c_2^2 |I - c_1|^{-1} \tag{6.1}$$

from whence for $\mu > 0$, it follows that

$$W = \frac{1}{2}\int \mu \, dI = \begin{cases} -c_2 (c_1 - I)^{1/2} + c_3, & \text{if } I < c_1 \\ c_2 (I - c_1)^{1/2} + c_3, & \text{if } I > c_1 \end{cases} \tag{6.2}$$

(6.3)

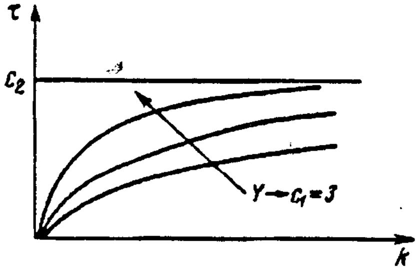

Fig. 11

where c_1, $c_2 > 0$, and c_3 are integration constants (parameters of the potential).

It is easy to verify that conditions (6.2) and (6.3) satisfy Eq. (1.23), both points I_- and I_+ lying at the left or right side from the point $I = c_1$. Consequently, the constructed potential (Fig.9) is fundamental. At the same time, $V(I_+, I) = I - c_1$ does not depend on I_+ while the PT region defined by inequalities (1.24), $0 < I - c_1 \le I - Y$, is the semizone

$$I > c_1, \quad 3 \le Y \le c_1 \tag{6.4}$$

(PT's are possible only at the branch (6.3)).

As follows from (3.3), (2.2), and (2.6),

$$\tau^2(I, Y) = c_2^2(I - Y)/|I - c_1| \tag{6.5}$$

The dependence of τ^2 on I for different relations between Y and c_1 is given in Fig.10. Obviously, inequality (3.10) also defines the semizone (6.4), which is at the same time, the nonellipticity region due to (4.7).

We assume that $c_1 = 3$. Then we have

$$W = c_2(I - 3)^{1/2} + c_3, \quad \tau = \mu k = c_2(k^2 + Y - 3)^{-1/2}k \tag{6.6}$$

The PT region contracts to the line:

$$Y = 3, I > 3 \tag{6.7}$$

where a material shows itself as a perfectly plastic (Fig.11): $\tau = c_2 \operatorname{sgn} k$. The six curved rays $\lambda_i = 1$ and $\lambda_j \lambda_k = 1$ on the ellipticity surface correspond to the line (6.7). According to (1.28) any two points at the ray may be connected by a two-dimensional jump with the normal $n_j^2 = (1 + \lambda_j^2)^{-1}$, $n_i = 0$.

Pasting, in any fashion, fragments of the fundamental potential into the convex dependences $W(I)$ it is possible to obtain different elastic plastic dependences $\tau(k)$.

REFERENCES

1. A. B. Freidin and A. M. Chiskis, "Regions of phase translation in nonlinear-elastic isotropic materials. 1. Basic relations," Izv. AN. MTT [Mechanics of Solids], vol. 29, no. 4, pp. 91-109, 1994.

2. M. A. Grinfeld, Methods of Continuum Mechanics in the Theory of Phase Transformations [in Russian], Nauka, Moscow, 1990.

3. R. Abeyaratne and J. K. Knowles, "Equilibrium shocks in plane deformations of incompressible elastic materials," J. Elasticity, vol. 22, no. 2, pp. 193-200, 1989.

4. A. I. Lur'e, Nonlinear Theory of Elasticity [in Russian], Nauka, Moscow, 1980.

5. R. Abeyaratne, "Discontinuous deformation gradients in plane finite elastostatics of incompressible materials," J. Elasticity, vol. 10, no. 3, pp. 255-293, 1980.

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