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DEFORMATIONS WITH DISCONTINUOUS GRADIENTS AND ELLIPTICITY OF ISOTROPIC HYPERELASTIC MATERIALS

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Abstract. *If a hyperelastic material is isotropic, the amplitude vector on the equilibrium surface of the deformation gradient discontinuity can be represented as an isotropic function of the strain tensor on one side of this surface and the normal to the surface. We analyze the consequences of this representation. Particular attention is given to the nontrivial zero jump solutions. The developed procedure for the zero jumps zone construction in the strain space is illustrated by the consideration of some compressible and incompressible materials. Correlation between the ellipticity at given deformations and nontrivial zero solutions is studied.*

1 INTRODUCTION

Ellipticity of elastic materials has been the subject of much investigation. (See e.g.[1–11] and references cited therein.) Among others one can note the following reasons to study ellipticity:

* The strong ellipticity requirements can be used in rejection of the constitutive equations in the sense that "a good material is an elliptic one".

* Strain localization, shear failures and crack problems in finite elastostatics are accompanied by a failure of ordinary of ellipticity at some deformations.[2,3]

* Deformations with discontinuous gradients play a key role in the current theories of phase transformations in elastic bodies. Such deformation field can exist only if ellipticity fails at some deformations[3,4,7,9,12]. The non-ellipticity zone of necessity crosses the phase transition zone formed by all deformations which can coexist at the phase boundary[13,14].

The last circumstances have a dominant role in our consideration. In Section 2 we recall the relevant notions of strong and ordinary ellipticity and introduce the concept of a zero jumps zone. In Section 3 we reformulate the jump conditions on the shock surface in isotropic compressible and incompressible materials in terms of deformation and *orientation* invariants[13] and give some preliminary analysis of admissible jumps. Section 4 is devoted to the consideration of small jumps of strains and the zero jumps zone construction in a space of deformation invariants. The problem is discussed in the context of the ellipticity examination.

2 ZERO JUMPS ZONE AND ELLIPTICITY

Let $\mathbf{F}$, $W(\mathbf{F})$ and $\mathcal{U}$ be the deformation gradient tensor, an elastic potential and a set of unit vectors respectively. An elastic material is said[1–7] to be ordinary elliptic at $\mathbf{F}$ if

$$\det \mathbf{Q}(\mathbf{F},\mathbf{N}) \neq 0 \quad \forall \mathbf{N} \in \mathcal{U} \tag{2.1}$$

and strongly elliptic at $\mathbf{F}$ if

$$\mathbf{e} \cdot \mathbf{Q}(\mathbf{F},\mathbf{N})\mathbf{e} > 0 \qquad \forall \mathbf{e}, \mathbf{N} \in \mathcal{U} \tag{2.2}$$

where $\mathbf{Q}(\mathbf{F},\mathbf{N})$ is the acoustic tensor. If $\mathbf{C}(\mathbf{F}) = W_{\mathbf{FF}}$ is the elasticity tensor, then $Q_{ik}(\mathbf{F},\mathbf{N}) = C_{ijkl}(\mathbf{F})\tilde{N}_j\tilde{N}_l$, $\mathbf{N} \in \mathcal{U}$. One can reformulate (2.2): the material is strong elliptic at $\mathbf{F}$ if and only if

$$\det \mathbf{Q}(\mathbf{F},\mathbf{N}) \neq 0 \quad \forall \mathbf{N} \in \mathcal{U}, \quad \exists \mathbf{N}_*: \quad Q_i(\mathbf{F},\mathbf{N}_*) > 0 \quad (i = 1,2,3) \tag{2.3}$$

where Q_i are the eigenvalues of $\mathbf{Q}$.

It is known[3,4,7,9,12], that a loss of strong ellipticity at some deformation is a necessary condition for the existence of equilibrium piecewise homogeneous strain fields with the surfaces of discontinuity in deformation gradients at continuous displacements.

We consider a surface Γ (a shock surface) of the discontinuity in the deformation gradient which satisfies the kinematic Hadamard condition

$$[\mathbf{F}] = \mathbf{f} \otimes \mathbf{N} \tag{2.4}$$

and the traction continuity condition

$$[\mathbf{S}]\,\mathbf{N} = 0 \tag{2.5}$$

where $\mathbf{S} = W_{\mathbf{F}}$ is the Piola stress tensor, $[\cdot] = (\cdot)^+ - (\cdot)^-$ denotes the jump of a function across Γ, indexes "$\pm$" identify the values on different sides of the shock surface, $\mathbf{N}$ and $\mathbf{f}$ are the unit normal on Γ and amplitude vector in the reference configuration. In view of (2.4) the traction continuity condition (2.5) takes the form

$$\left(\mathbf{S}\left(\mathbf{F} + \mathbf{f} \otimes \mathbf{N}\right) - \mathbf{S}\left(\mathbf{F}\right)\right)\mathbf{N} = 0 \tag{2.6}$$

and can be considered as the equation in $\mathbf{f}$ at given $\mathbf{F} = \mathbf{F}_-$ and $\mathbf{N}$ or the equation to determine a two–parameter family of shocks $\{\mathbf{f}, \mathbf{N}\}$ at given $\mathbf{F}$. This equation has the trivial solution $\mathbf{f} = 0$ at any $\mathbf{F}$ and $\mathbf{N}$.

The condition (2.6) can be satisfied at $\mathbf{f} \neq \mathbf{0}$ only if[3,12] on the segment $\mathbf{F}\left(\eta\right) = \mathbf{F}_- + \eta \mathbf{f} \otimes \mathbf{N}$ ($\eta \in [0, 1]$), which connects $\mathbf{F}_-$ and $\mathbf{F}_+$,

$$\exists\, \mathbf{F}_0 = \mathbf{F}(\eta_0): \quad \mathbf{f} \cdot \mathbf{Q}\left(\mathbf{F}_0, \mathbf{N}\right)\mathbf{f} = 0$$

Note that the material can be strongly elliptic at coexisting gradients $\mathbf{F}_\pm$ (see e.g.[7,13,14]).

Now suppose that $\mathbf{f} = \varepsilon \mathbf{e}$, where $\mathbf{e} \in \mathcal{U}$, $0 < \varepsilon \ll 1$. If $W\left(\mathbf{F}\right)$ is smooth enough, equation (2.6) takes the form

$$\varepsilon\left(\mathbf{Q}\left(\mathbf{F}_0, \mathbf{N}\right)\mathbf{e} + o\left(\varepsilon\right)\right) = 0 \tag{2.7}$$

At $\varepsilon \neq 0$, $\varepsilon \to 0$ it follows from (2.7) that $\mathbf{Q}\left(\mathbf{F}_0, \mathbf{N}\right)\mathbf{e} \to \mathbf{0}$. Consequently, if (2.6) has nontrivial zero solution at $\mathbf{F}$, then

$$\exists\, \mathbf{N}, \mathbf{e} \in \mathcal{U}: \quad \mathbf{Q}\left(\mathbf{F}, \mathbf{N}\right)\mathbf{e} = \mathbf{0} \tag{2.8}$$

which gives

$$\exists\, \mathbf{N} \in \mathcal{U}: \quad \det\ \mathbf{Q}\left(\mathbf{F}, \mathbf{N}\right) = 0 \tag{2.9}$$

At given $\mathbf{F}$ the equation (2.9) can be considered as the equation for the one–parametric family of unit normals $\mathbf{N}$ assisted by the one–parametric family of the vectors $\mathbf{e}$.

Definition. *Deformation gradients F at which the equation* (2.6) *besides trivial has only nontrivial zero solutions form the zero jumps zone $\mathcal{F}_0$.*

Further we develop a procedure for the zero jumps zone construction in a case of isotropic material. It should be stressed that the loss of ordinary ellipticity is a necessary condition providing the existence of nontrivial zero jumps and we will demonstrate finite jumps which are also possible if (2.9) is met. But these jumps are allowed at given values of deformation invariants if and only if at least one of the Baker–Ericksen inequalities degenerates into the equality. Note that checking of the Baker–Ericksen inequalities as well as the developed procedure for the zero jumps zone construction seem to be more simple then the direct examination of det $\mathbf{Q}(\mathbf{F}, \mathbf{N})$. Now we do not formulate the restrictions on elastic potentials at which the other finite jumps are not allowed by (2.9), but the direct consideration of some classes of elastic materials shows that the examination of strong ellipticity can be reduced to the construction of the zero jumps zones. Note also that the zero jumps surface is a part of the phase transition zone[13,14] and, consequently, is of interest irrespective of ellipticity.

3 JUMP CONDITIONS IN TERMS OF DEFORMATION AND ORIENTATION INVARIANTS

The conditions (2.4), (2.5) can be rewritten as

$$\mathbf{F}_+ - \mathbf{F}_- = \mathbf{a}_\mp \otimes \mathbf{n}\mathbf{F}_\mp, \qquad [\mathbf{T}]\mathbf{n} = 0 \tag{3.1}$$

where $\mathbf{n} \in \mathcal{U}$ and $\mathbf{a}_\mp = |\mathbf{F}_\mp^T \mathbf{n}|^{-1}\mathbf{f}$ are the normal to the shock surface and amplitude in the actual configuration, $\mathbf{T}$ is the Cauchy stress tensor.

Let $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy–Green deformation tensor, $\mathbf{e}_i$ and $\lambda_i > 0$ $(i = 1, 2, 3)$ are the eigenvectors of $\mathbf{B}$ and principal stretches, $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2$, $J = \lambda_1\lambda_2\lambda_3$ are the deformation invariants. We also introduce the *orientation invariants*[13]

$$N_k = \mathbf{n} \cdot \mathbf{B}^k \mathbf{n} \qquad k = \pm 1, \pm 2, ... \tag{3.2}$$

Due to the Caley–Hamilton theorem only two such invariants are independent. At given $\mathbf{B}$ a couple of orientation invariants (N_s, N_t) $(t \neq s)$ determines the normal $\mathbf{n}$ by the system of equations

$$\sum n_i^2 = 1, \qquad \sum n_i^2 \lambda_i^{2k} = N_k \quad (k = s, t), \qquad n_i = \mathbf{n} \cdot \mathbf{e}_i \tag{3.3}$$

which is linear with respect to n_i^2 $(i = 1, 2, 3)$.

Since the solution of the system (3.3) has to be non–negative, the admissible values domain $\mathcal{D}_{st}$ for orientation invariants N_s, N_t is a triangle with vertexes $(\lambda_i^{2s}, \lambda_i^{2t})$ $(i = 1, 2, 3)$ lying on the *skeleton curve* $N_s = N_t^{s/t}$. The vertexes $(\lambda_i^{2s}, \lambda_i^{2t})$ correspond to $\mathbf{n} = \mathbf{e}_i$, points of the $i - j$ – side of the triangle correspond to $\mathbf{n}$ lying in the $i - j$ –principal plane of $\mathbf{B}$. The triangle $\mathcal{D}_{st}$ degenerates into the segment or the point if $\lambda_i = \lambda_j \neq \lambda_k$ or $\lambda_1 = \lambda_2 = \lambda_3$ respectively.

Using $(3.1)_1$ one can arrive the following relations[13] between the deformation invariants on different sides of the shock surface:

$$\begin{aligned} \frac{J_+}{J_-} &= 1 + \mathbf{a}\cdot\mathbf{n} && (3.4)\\ [I_1] &= 2\mathbf{a}\cdot\mathbf{B}_-\mathbf{n} + N_1\mathbf{a}\cdot\mathbf{a} && (3.5)\\ \left[\frac{I_2}{J^2}\right] &= \left(\frac{J_-}{J_+}\right)^2 \mathbf{a}\cdot\mathbf{B}_-^{-1}\mathbf{a} - 2\frac{J_-}{J_+}\mathbf{a}\cdot\mathbf{B}_-^{-1}\mathbf{n} && (3.6) \end{aligned}$$

where $\mathbf{a} = \mathbf{a}_-$. It follows from (3.4)

$$\mathbf{a} = \left(1 - \frac{J_+}{J_-}\right)\mathbf{n} + \pi \qquad (3.7)$$

where $\pi = \mathbf{Pa}$, $\mathbf{E}$ and $\mathbf{P} = \mathbf{E} - \mathbf{n}\otimes\mathbf{n}$ are the unit tensor and projector.

If the material is isotropic, then

$$\begin{aligned} W &= W(I_1, I_2, J), \quad \mathbf{T} = \mu_0\mathbf{E} + \mu_1\mathbf{B} + \mu_{-1}\mathbf{B}^{-1}\\ \mu_0 &= W_3 + 2J^{-1}I_2W_2, \ \mu_1 = 2J^{-1}W_1, \ \mu_{-1} = 2JW_2 \end{aligned}$$

where W_1, W_2, W_3 denote $\partial W/\partial I_1$, $\partial W/\partial I_2$ and $\partial W/\partial J$ respectively.

Projecting the traction vector $\mathbf{Tn}$ onto the normal $\mathbf{n}$ and plane tangent to the shock surface we arrive

$$\begin{aligned} [\mu_0] + [\mu_1 N_1] + [\mu_{-1}N_{-1}] &= 0 && (3.8)\\ [\mu_1\mathbf{PBn}] + [\mu_{-1}\mathbf{PB}^{-1}\mathbf{n}] &= 0 && (3.9) \end{aligned}$$

Equation (3.8) can be rewritten as

$$-\frac{1}{2}[W_3] = [JW_1]\frac{N_1}{J^2} + [JW_2]\left(\frac{I_2}{J^2} - N_{-1}\right) \qquad (3.10)$$

Notice that

$$\left[\frac{N_1}{J^2}\right] = 0, \quad \left[\frac{I_2}{J^2} - N_{-1}\right] = 0 \qquad (3.11)$$

due to the kinematic condition. So we do not show indexes "±" when write $\frac{N_1}{J^2}$ and $\left(\frac{I_2}{J^2} - N_{-1}\right)$.

In a case of incompressible materials $W = W(I_1, I_2)$

$$\mathbf{T} = -p\mathbf{E} + \mu_1\mathbf{B} + \mu_{-1}\mathbf{B}^{-1} \qquad \mu_1 = 2W_1, \ \mu_{-1} = 2W_2$$

The condition for the normal component of the traction determine the jump of the reaction p in dependence on deformation and orientation invariants

$$[p] = 2\,[W_1]\,N_1 + [W_2]\,(I_2 - N_{-1}) \tag{3.12}$$

The equation (3.9) for the tangent component of the traction takes the form

$$W_1^+ N_1^- \pi + J_-^2 W_2^+ \mathbf{P}\mathbf{B}_-^{-1}\pi = -\,[W_1]\,\mathbf{P}\mathbf{B}_-\mathbf{n} + \mathbf{J}_-^2\,[W_2]\,\mathbf{P}\mathbf{B}_-^{-1}\mathbf{n} \tag{3.13}$$

and leads to the following representative

Theorem.[13] *Assume that the material is isotropic and on the shock surface*

$$\frac{N_1}{J^2}\left(W_1^+\right)^2 + \left(\frac{I_2}{J^2} - N_{-1}\right) W_1^+ W_2^+ + \left(W_2^+\right)^2 \neq 0 \tag{3.14}$$

Then the amplitude $\mathbf{a}$ *can be uniquely presented as the isotropic function of the strain tensor* $\mathbf{B}_-$ *and the normal* $\mathbf{n}$

$$\mathbf{a} = \gamma\mathbf{n} + \alpha\mathbf{B}_-\mathbf{n} + \beta\mathbf{B}_-^{-1}\mathbf{n} \tag{3.15}$$

moreover

$$\mathbf{a} = \left(\frac{J_+}{J_-} - 1\right)\mathbf{n} + \alpha\mathbf{t}_1 + \beta\mathbf{t}_{-1} \tag{3.16}$$

where $\mathbf{t}_1 = \mathbf{P}\mathbf{B}_-\mathbf{n}$, $\mathbf{t}_{-1} = \mathbf{P}\mathbf{B}_-^{-1}\mathbf{n}$, *the coefficients* α *and* β *are the functions of the deformation and orientation invariants and can be evaluated from the system of linear equations*

$$\begin{pmatrix} N_1^- W_1^+ & W_2^+ \\ -N_1^- W_2^+ & \frac{N_1}{J^2} W_1^+ + \left(\frac{I_2}{J^2} - N_{-1}\right) W_2^+ \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\,[W_1] \\ [W_2] \end{pmatrix} \quad \textit{if} \quad n_1 n_2 n_3 \neq 0 \tag{3.17}$$

or

$$\alpha = \frac{[W_1] + \lambda_{k+}^2\,[W_2]}{N_1\left(W_1^+ + \lambda_{k+}^2 W_2^+\right)}, \quad \beta = 0 \quad \textit{if}\ n_k = 0,\ n_i n_j \neq 0,\ \lambda_i \neq \lambda_j \tag{3.18}$$

The theorem remains true for the incompressible materials, one needs only to set $J \equiv 1$.

Note that (3.16) means that $\pi = \alpha\mathbf{t}_1 + \beta\mathbf{t}_{-1}$ where $\mathbf{t}_1 = \mathbf{t}_{-1} = 0$ if and only if one of the following conditions is met:

$$(a)\ \ \mathbf{n} = \mathbf{e}_i \qquad (b)\ \ n_k = 0,\ n_i n_j \neq 0,\ \lambda_{i-} = \lambda_{j-} \qquad (c)\ \ \mathbf{B}_- = \lambda^2\mathbf{E} \tag{3.19}$$

In these cases $\pi = 0$. For brevity we will not dwell on (b) and (c). Another degeneracy of the basis $(\mathbf{t}_1, \mathbf{t}_{-1})$ is $\mathbf{t}_1 \parallel \mathbf{t}_{-1} \neq 0$ that happens if and only if $n_k = 0,\ n_i n_j \neq 0,\ \ \lambda_i \neq \lambda_j$. In the latter case

$$\mathbf{t}_1 = \left(\lambda_{i-}^2 - \lambda_{j-}^2\right) n_i n_j \left(n_j \mathbf{e}_i - n_i \mathbf{e}_j\right), \quad \mathbf{t}_{-1} = -\lambda_{i-}^{-2}\lambda_{j-}^{-2}\mathbf{t}_1 \tag{3.20}$$

and the vectors $\pi = (\alpha - \beta\lambda_{i-}^{-2}\lambda_{j-}^{-2})\mathbf{t}_1$ and $\mathbf{n}$ lie in the $i-j$-principal plane of $\mathbf{B}_-$ (plane jumps); in (3.18) we renamed $\alpha - \beta\lambda_{i-}^{-2}\lambda_{j-}^{-2} \to \alpha$.

Among the consequences of (3.16) – (3.18) may be listed the following ones:

$$W_1 = const,\ W_2 = const \Rightarrow \pi = 0, \quad \mathbf{a} = \left(\frac{J_+}{J_-} - 1\right)\mathbf{n} \tag{3.21}$$

$$W = W(I_\alpha, J) \ \ (\alpha = 1 \ or \ 2) \Rightarrow \mathbf{a} = \left(\frac{J_+}{J_-} - 1\right)\mathbf{n} - \frac{[W_\alpha]}{W_\alpha^+ N_1^-}\mathbf{P}\mathbf{B}_-\mathbf{n} \tag{3.22}$$

The representation (3.15) has been used in the construction of phase transition zones[13,14] (an additional thermodynamic condition was added to the conditions (2.4), (2.5)). In this paper we study discontinuous solutions irrespective of the thermodynamic condition.

The function

$$D = \frac{N_1^+}{J_+^2}\left(W_1^+\right)^2 + \left(\frac{I_2^+}{J_+^2} - N_{-1}^+\right) W_1^+ W_2^+ + \left(W_2^+\right)^2$$

is linear in N_1^+, N_{-1}^+. On the $i-j$–side of the triangle $\mathcal{D}_{1-1}^+$

$$\frac{I_2^+}{J_+^2} - N_{-1}^+ = \frac{J_+^2}{\lambda_{k+}^2} + \lambda_{k+}^2 N_1^+, \qquad N_1 \in \left[\lambda_i^2, \lambda_j^2\right] \tag{3.23}$$

$$D = J_+^{-2}\left(W_1^+ + \lambda_{k+}^2 W_2^+\right)\left(N_1^+ W_1^+ + \lambda_{k+}^{-2} J_+^2 W_2^+\right)$$

Maxima and minima values of D are reached at the vertexes of the triangle $\mathcal{D}_{1-1}$ and are among the values $\lambda_{i+}^2 J_+^{-2}\left(W_1^+ + \lambda_{k+}^2 W_2^+\right)\left(W_1^+ + \lambda_{j+}^2 W_2^+\right)$. Thus at given I_1^+, I_2^+, J_+, $D \neq 0 \ \ \forall N_1^+, N_{-1}^+ \in \mathcal{D}_{1-1}^+$ if and only if

$$W_1^+ + \lambda_{k+}^2 W_2^+ > 0 \ \ (k = 1,2,3) \quad or \quad W_1^+ + \lambda_{k+}^2 W_2^+ < 0 \ \ (k = 1,2,3) \tag{3.24}$$

Recall that if the material is strongly elliptic at I_1^+, I_2^+, J_+, then $W_1^+ + \lambda_{i+}^2 W_2^+ > 0$ $(i = 1,2,3)$ (the Baker–Ericksen inequalities). It follows from (3.24) that if

$$\exists\, N_1^+, N_{-1}^+ \in \mathcal{D}_{1-1}^+ : \ D\left(N_1^+, N_{-1}^+ \,|\, I_1^+, I_2^+, J_+\right) = 0 \tag{3.25}$$

then among $\lambda_{i+}\ (i = 1,2,3)$ there exists $\lambda_{k+} : W_1^+ + \lambda_{k+}^2 W_2^+ \leq 0$, thus ordinary (and strong) ellipticity fails. The traction condition (3.13) is satisfied at $\forall\pi$ if $D = 0$. Note, that the equality $D = 0$ is compatible with the conditions (3.10), (3.13) if the additional restrictions on the derivatives of $W(I_1, I_2, J)$ are satisfied.

Kinematic relations (3.4) – (3.6) give $[I_1] = [I_2] = [J] = 0$ if $\mathbf{a} = 0$. On the other hand, if $[I_1] = [I_2] = [J] = 0$, then (3.4), (3.5) give

$$\mathbf{a} = \pi = -\,2N_1^{-1}\mathbf{t}_1 \tag{3.26}$$

One can check that conditions (3.6) and (3.10) are satisfied. The condition (3.13) takes the form

$$W_1\mathbf{t}_1 - J_-^2 W_2 \mathbf{t}_{-1} = 0 \tag{3.27}$$

In the cases listed in (3.19), $\mathbf{a} = 0$ and (3.27) is fulfilled because of $\mathbf{t}_1 = \mathbf{t}_{-1} = 0$. In the other cases it follows from (3.27)and (3.20) that jumps at which

$$[I_1] = [I_2] = [J] = 0 \quad \text{and} \quad \mathbf{a} \neq 0 \tag{3.28}$$

satisfies kinematic and traction conditions only if at I_1, I_2, J

$$\begin{array}{ll} W_1 = W_2 = 0 & (n_1 n_2 n_3 \neq 0) \\ W_1 + \lambda_k^2 W_2 = 0 & (n_k = 0,\ n_i n_j \neq 0,\ \lambda_i \neq \lambda_j) \end{array} \tag{3.29}$$

Thus, these jumps are allowed only if ordinary ellipticity fails at I_1, I_2, J and are impossible if $D \neq 0$. The above consideration demonstrates the shocks for which the deformation invariants are continuous, despite that fact that deformation gradient is discontinuous.

In plane elastostatics relationships (3.29) determine a line on the (λ_i, λ_j)–plane at fixed λ_k. A shear modulus is zero on this line. It can be shown that deformations on the different sides of the shock surface differ only by the sign of the shear parameter $k = |\lambda_i - \lambda_j|$. Thus we reproduce the result presented by Rosakis[9].

We emphasize that the solution (3.28) as well as the equality $D = 0$ are not allowed if the Baker–Ericksen inequalities are satisfied at given I_1, I_2, J.

Assume $D \neq 0 \ \ \forall N_1^+, N_{-1}^+ \in \mathcal{D}_{1-1}^+$. Then after the substitution (3.16) into (3.5), (3.6) we obtain the system of five equations which are (3.10), (3.17) and transformed (3.5) and (3.6). These equations can be considered as the system to determine seven unknowns I_1^+, I_2^+, J_+, α, β, N_1^- and N_{-1}^-as the functions of I_1^-, I_2^-, J_-. The solution of this system, if exists, gives two–parameter family of jumps.

In a case of incompressible material the equation (3.12) is split and determine two–parameter family of $[p]$.

In plane elastostatics after the substitution (3.16), (3.18) and (3.23) into (3.5), (3.6) we arrive the system of three equations ((3.10) is added) for four unknowns I_1^+, I_2^+, J_+ and N_1^-. The solution gives one–parameter family of jumps.

4 ZERO JUMP LINES AND ELLIPTICITY

With the small jumps of strains in mind one can linearize the above mentioned system of equations and arrive the linear system of five equations for five unknowns $[I_1]$, $[I_2]$, $[J]$, α and β

$$\begin{gathered}
\frac{N_1}{J}[J] - \frac{1}{2}[I_1] + \alpha\left(N_2 - N_1^2\right) + \beta\left(1 - N_1 N_{-1}\right) = 0 \\
-\frac{1}{J}\left(\frac{I_2}{J^2} - N_{-1}\right)[J] + \frac{1}{2J^2}[I_2] + \alpha\left(1 - N_1 N_{-1}\right) + \beta\left(N_{-2} - N_{-1}^2\right) = 0 \\
W_{13}[J] + W_{11}[I_1] + W_{12}[I_2] + N_1 W_1 \alpha + W_2 \beta = 0 \\
W_{23}[J] + W_{12}[I_1] + W_{22}[I_2] + N_1 W_2 \alpha - \left(\frac{N_1}{J^2} W_1 + \left(\frac{I_2}{J^2} - N_{-1}\right) W_2\right)\beta = 0 \\
\left(\frac{N_1}{J^2}\left(W_1 + J W_{13}\right) + \left(\frac{I_2}{J^2} - N_{-1}\right)\left(W_2 + J W_{23}\right) + \frac{1}{2} W_{33}\right)[J] + \\
\left(\frac{N_1}{J} W_{11} + \left(\frac{I_2}{J^2} - N_{-1}\right) J W_{12} + \frac{1}{2} W_{13}\right)[I_1] + \\
\left(\frac{N_1}{J} W_{12} + \left(\frac{I_2}{J^2} - N_{-1}\right) J W_{22} + \frac{1}{2} W_{23}\right)[I_2] = 0
\end{gathered} \tag{4.1}$$

Recall that among the orientation invariants N_2, N_1, N_{-1}, N_{-2} only two of them are independent. We assume $n_1 n_2 n_3 \neq 0$. The other cases can be considered analogously.

We imply that $D \neq 0$ and, consequently,

$$\mathbf{a} \to 0 \Longleftrightarrow \{[I_1] \to 0,\ [I_2] \to 0,\ [J] \to 0\}$$

At given I_1, I_2, J the system (3.18) has only the trivial solution $[I_1] = [I_2] = [J] = \alpha = \beta = 0$ if and only if

$$H(I_1, I_2, J, N_s, N_t) \neq 0 \qquad \forall (N_s, N_t) \in \mathcal{D}_{st}$$

where H is the determinant of the system (3.18). Then nontrivial zero solutions correspond to the *zero jump lines* on the N_s, N_t – plane:

$$H(N_s, N_t \mid I_1, I_2, J) = 0 \qquad (N_s, N_t) \in \mathcal{D}_{st}$$

The zero jumps zone in the space of deformation invariants is determined by the system of inequalities

$$\begin{gathered}
H_{\min}(I_1, I_2, J) \leq 0 \leq H_{\max}(I_1, I_2, J) \\
W_1 + \lambda_k^2 W_2 > 0 \ (k = 1,2,3) \quad or \quad W_1 + \lambda_k^2 W_2 < 0 \ (k = 1,2,3) \\
H_{\min}(I_1, I_2, J) = \min_{(N_s, N_t) \in \mathcal{D}_{st}} H(N_s, N_t \mid I_1, I_2, J) \\
H_{\max}(I_1, I_2, J) = \max_{(N_s, N_t) \in \mathcal{D}_{st}} H(N_s, N_t \mid I_1, I_2, J)
\end{gathered} \tag{4.2}$$

The zero jump zone construction is reduced to the investigation of the polynomial function of two variables N_s, N_t. Now we can formulate

Proposition. *If the material is strongly elliptic at* I_1, I_2, J, *then*

$$H_{\min}(I_1, I_2, J) > 0 \quad or \quad H_{\max}(I_1, I_2, J) < 0 \tag{4.3}$$

$$W_1 + \lambda_{\min}^2 W_2 > 0, \; W_1 + \lambda_{\max}^2 W_2 > 0 \tag{4.4}$$

$$2\left(W_1 + \left(I_1 - \lambda_{\mathrm{md}}^2\right) W_2\right) + 4\lambda_{\mathrm{md}}^2 \left(W_{11} + 2\left(I_1 - \lambda_{\mathrm{md}}^2\right) W_{12} + \left(I_1 - \lambda_{\mathrm{md}}^2\right)^2 W_{22}\right) + J^2 \lambda_{\mathrm{md}}^2 W_{33} + J\left(W_{13} + \left(I_1 - \lambda_{\mathrm{md}}^2\right) W_{23}\right) > 0 \tag{4.5}$$

where $\lambda_{\min}$, λ_{md} *and* $\lambda_{\max}$ *are the minimal, intermediate and maximal values of the principal stretches.*

The relationship (4.5) implies that $\partial t_{\mathrm{md}}/\partial \lambda_{\mathrm{md}} > 0$, where t_{md} is the principal Cauchy stress corresponding to the eigenvector $\mathbf{e}_{\mathrm{md}}$ of the tensor $\mathbf{B}$.

If the loss of ordinary ellipticity is also a sufficient condition for the nontrivial zero jumps existence or the equality $D = 0$, then "if" in the above Proposition can be replaced by "if and only if".

To illustrate the above ideas we considered some known classes of compressible and incompressible elastic materials. Below is given one of the examples.

Hadamard materials are materials with an elastic potential given by

$$W(I_1, I_2, J) = \frac{c}{2} I_1 + \frac{d}{2} I_2 + \Phi(J) \tag{4.6}$$

where c and d are material constants, Φ is a function twice continuously differentiable on $(0, \infty)$. The inequalities (4.4), (4.5) produce

$$c + d\lambda_{\min}^2 > 0, \;\; c + d\lambda_{\max}^2 > 0, \;\; c + d\left(\lambda_{\min}^2 + \lambda_{\max}^2\right) + \lambda_{\min}^2 \lambda_{\max}^2 \Phi'' > 0 \tag{4.7}$$

where the prime denotes differentiation with respect to the argument.

The maxima and minima values of D are among the values

$$\lambda_k^{-2} \lambda_i^{-2} \left(c + d\lambda_k^2\right)\left(c + d\lambda_i^2\right) \qquad i, k = 1, 2, 3 \;\; (i \neq k)$$

and are positive because of (4.7). Due to (3.21) the Hadamard material allows only normal shocks with the amplitude $\mathbf{a} = \left(\frac{J_+}{J_-} - 1\right) \mathbf{n}$, $\mathbf{a} = 0 \Longleftrightarrow [J] = 0$.

The system (4.1) takes the form

$$\frac{N_1}{J}[J] - \frac{1}{2}[I_1] = 0, \quad -\left(\frac{I_2}{J^2} - N_{-1}\right)[J] + \frac{1}{2J}[I_2] = 0 \tag{4.8}$$

$$\left(\frac{N_1}{J^2} c + \left(\frac{I_2}{J^2} - N_{-1}\right) d + \Phi''\right)[J] = 0 \tag{4.9}$$

Since

$$\min_{(N_1,N_{-1})\in\mathcal{D}_{1-1}}\left(\frac{N_1}{J^2}c+\left(\frac{I_2}{J^2}-N_{-1}\right)d\right)=h\left(\lambda_{\max},\lambda_{\mathrm{md}}\right) \qquad (4.10)$$
$$\max_{(N_1,N_{-1})\in\mathcal{D}_{1-1}}\left(\frac{N_1}{J^2}c+\left(\frac{I_2}{J^2}-N_{-1}\right)d\right)=h(\lambda_{\min},\lambda_{\mathrm{md}})$$

where $h\left(\lambda_i,\lambda_j\right)\overset{def}{=}c\lambda_i^{-2}\lambda_j^{-2}+d\left(\lambda_i^{-2}+\lambda_j^{-2}\right)$, the non zero solution exists only if

$$\Phi''\left(J\right)+h\left(\lambda_{\max},\lambda_{\mathrm{md}}\right)>0 \quad or \quad \Phi''\left(J\right)+\ h(\lambda_{\min},\lambda_{\mathrm{md}})<0 \qquad (4.11)$$

¿From the inequalities (4.7) and (4.11) it follows that if strong ellipticity holds at I_1, I_2, J, then

$$c+d\lambda_{\min}^2>0,\ \ c+d\lambda_{\max}^2>0,\ \ c\lambda_{\max}^{-2}\lambda_{\mathrm{md}}^{-2}+d\left(\lambda_{\max}^{-2}+\lambda_{\mathrm{md}}^{-2}\right)+\Phi''\left(J\right)>0 \qquad (4.12)$$

In[7] it is proved that the conditions (4.12) are also the sufficient conditions.

Note that irrespectively of ellipticity and smallness of $[J]$, the Poisson condition (3.10) for the normal component of traction gives

$$\frac{[\Phi']}{[J]}=-\left(\frac{N_1}{J^2}c+\left(\frac{I_2}{J^2}-N_{-1}\right)d\right),\ \ h\left(\lambda_{\max},\lambda_{\mathrm{md}}\right)\leq-\frac{[\Phi']}{[J]}\leq h(\lambda_{\min},\lambda_{\mathrm{md}}) \qquad (4.13)$$

The relationships (4.12), (4.13) confirm that the material allows shock solution only if ellipticity fails at some deformations. In addition, it follows from (4.13), that if $c\geq 0$, $d\geq 0$, $c+d\neq 0$ and $J_\pm$ correspond to the coexistent deformations, then $[\Phi']\,/\,[J]<0$. The above results were arrived[13] in the context of an examination of the phase transition zones and non-ellipticity zones construction. The same inequalities were obtained by Rosakis[7] at another approach.

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