

On phase transformations of an inclusion in an external strain field

Elena N. Vilchevskaya

Alexander B. Freidin

ven@itcwin.com freidin@mechanics.ipme.ru

Abstract

Solid materials exhibiting phase transformations under the process of deformation have received increased attention in recent years. This reflects the tendency of modern mechanics to expand on the process of structures formation and changes. Stress-induced phase transformations lead to martensite transformations in alloys, localized orientation transformations in polymers, polymorphous transitions in minerals, rock melting or solidification, freezing of soils. On the whole, these studies are directed, on one hand, towards the further understanding of material structure formation on different length scales depending on the loading path, and, on the other hand, towards the development and practical use of new (smart) materials and elements of structures that predictably and nontrivially respond to external actions.

Materials may contain inclusions that can endure phase transitions under the process of deformation. Favorably oriented anisotropic grains in polycrystalline material or material nonhomogeneities of various nature may be such inclusions. Phase transitions change elastic modulus of a material and produce self-strains of a transformation. As a result local stress-strain fields are changed. This in turn may initiate or block fracture processes.

We study phase transitions in a cylindrical inclusion under a homogenous external stress field transmitted by a linear elastic matrix. Since a material of the inclusion allows phase transformations, its free energy density is to be noconvex and is modeled by piece-wise quadratic dependencies of a linear strain tensor. The transition from one state to another is determined by the energy preferences. In strain-space we construct domains where one-phase states exist and then construct the transformation surface that determines when the first one-phase state of an inclusion is changed by another one. Then we study equilibrium two-phase states of an inclusion on the various loading paths.

1 Phase transformation of a cylindrical inclusion in an unbounded media.

Let us consider an unbounded linear-elastic media containing a cylindrical inclusion which can suffer a phase transitions under a homogenous external stress field ε_0 .

Since a material of the inclusion allows phase transformations, its free energy density is to be noconvex and is modeled by piece-wise quadratic dependencies of ε . We consider the two-branches dependence.

$$f(\varepsilon, \theta) = \min_{-,+} \{f_-(\varepsilon, \theta), f_+(\varepsilon, \theta)\}$$
$$f_{\pm}(\varepsilon, \theta) = f_{\pm}^0(\theta) + \frac{1}{2}(\varepsilon - \varepsilon_{\pm}^p) : \mathbf{C}_{\pm} : (\varepsilon - \varepsilon_{\pm}^p), \quad (1)$$

here \mathbf{C}_{\pm} are the positive definite tensors of the elastic moduli of the phases, f_{\pm}^0 , ε_{\pm}^p are free energy densities and strain tensors in unstressed phases "±".

Then constitutive equations take the form

$$\sigma_+(\varepsilon) = \mathbf{C}_+ : (\varepsilon - \varepsilon^p), \quad \sigma_-(\varepsilon) = \mathbf{C}_- : \varepsilon, \quad \sigma_0(\varepsilon) = \mathbf{C}_0 : \varepsilon, \quad (2)$$

here \mathbf{C}_0 – tensor of the elastic moduli of the matrix, (assume that $\varepsilon_-^p = 0$ in (1) then $\varepsilon_+^p \equiv \varepsilon^p$ is a phase transition self-strain tensor).

Let the inclusion suffers phase transition under some external field. We suppose now that the transition from one state to another is determined by the energy preferences. For energies comparison the difference of potential energies $\Pi_+ - \Pi_0$ and $\Pi_- - \Pi_0$ should be considered (Π_0 — the potential energy of the medium without an inclusion). It can be shown [1], [2]

$$\Pi_- - \Pi_0 = \frac{1}{2} \int_V \mathbf{q}_- : (\mathbf{C}_1^-)^{-1} : \mathbf{q}_- dV$$
$$\Pi_+ - \Pi_0 = \int_V \left(\gamma_*^+ + \frac{1}{2} \mathbf{q}_+ : (\mathbf{C}_1^+)^{-1} : \mathbf{q}_+ \right) dV, \quad (3)$$

where

$$\begin{aligned}
\mathbf{q}_0^- &= -\mathbf{C}_1^- : \boldsymbol{\varepsilon}_0, & \mathbf{q}_- &= -\mathbf{C}_1^- : \boldsymbol{\varepsilon}_-, \\
\mathbf{q}_0^+ &= \mathbf{C}_+ : \boldsymbol{\varepsilon}^p - \mathbf{C}_1^+ : \boldsymbol{\varepsilon}_0, & \mathbf{q}_+ &= \mathbf{C}_+ : \boldsymbol{\varepsilon}^p - \mathbf{C}_1^+ : \boldsymbol{\varepsilon}_+ \\
\mathbf{C}_1^- &= \mathbf{C}_- - \mathbf{C}_0, & \mathbf{C}_1^+ &= \mathbf{C}_+ - \mathbf{C}_0.
\end{aligned} \tag{4}$$

$\boldsymbol{\varepsilon}_-$, $\boldsymbol{\varepsilon}_+$ – the strains inside the inclusion in the states “-” “+” correspondingly, V – cross section of the inclusion. γ_*^+ is defined by :

$$\gamma_*^+ = [f^0] + \frac{1}{2}[\boldsymbol{\varepsilon}^p] : (\mathbf{C}_+^{-1} - \mathbf{C}_0^{-1})^{-1} : [\boldsymbol{\varepsilon}^p] \tag{5}$$

If $\boldsymbol{\varepsilon}_0 = \text{const}$ then the strain field inside the inclusion is constant too and can be written as [3],

$$\boldsymbol{\varepsilon}_{\pm} = \boldsymbol{\varepsilon}_0 + \mathbf{A} : \mathbf{q}_0^{\pm}$$

It follows that

$$\begin{aligned}
\boldsymbol{\varepsilon}_- &= \boldsymbol{\Lambda}_- : \boldsymbol{\varepsilon}_0^0, & \boldsymbol{\varepsilon}_+ &= \boldsymbol{\Lambda}_+ : (\boldsymbol{\varepsilon}_0 + \mathbf{A} : \mathbf{C}_+ : \boldsymbol{\varepsilon}^p) \\
\boldsymbol{\Lambda}_{\pm} &= (\mathbf{I} + \mathbf{A} : \mathbf{C}_1^{\pm})^{-1}
\end{aligned} \tag{6}$$

here \mathbf{I} – is a unit four-order tensor, \mathbf{A} depends on the geometry of the inclusion and matrix elastic moduli. For the circular cylinder in the isotropic matrix \mathbf{A} has the follow form: [3]

$$\begin{aligned}
\mathbf{A} &= \frac{1}{4\mu_0} \left[(1 - \alpha_0) \mathbf{P}^2 - (2 - \alpha_0) \left(\mathbf{P}^1 - \frac{1}{2} \mathbf{P}^2 \right) + 2\mathbf{P}^5 \right] \\
\alpha_0 &= \frac{1}{2(1 - \nu_0)}
\end{aligned} \tag{7}$$

here μ_0 and ν_0 – are shear modulus and the Poisson ration, the tensors \mathbf{P} are defined

$$\mathbf{P}_{\alpha\beta\lambda\mu}^1 = \mathbf{E}_{(\alpha)(\lambda)} \mathbf{E}_{(\mu)(\beta)}, \quad \mathbf{P}_{\alpha\beta\lambda\mu}^2 = \mathbf{E}_{\alpha\beta} \mathbf{E}_{\lambda\mu}, \quad \mathbf{P}_{\alpha\beta\lambda\mu}^5 = \mathbf{m}_{(\alpha} \mathbf{E}_{\beta)(\lambda)} \mathbf{m}_{\mu)}$$

$\mathbf{E}_{\alpha\beta} = \delta_{\alpha\beta}$, ($\alpha, \beta = 1, 2$) – a plane unit matrix, \mathbf{m} – a cylinder axial ort.

Let the phases “+” and “-” are isotropic

$$\mathbf{C}_{\pm} = \tilde{k}_{\pm} \mathbf{E} \mathbf{E} + 2\mu_{\pm} \left(\mathbf{I} - \frac{1}{2} \mathbf{E} \mathbf{E} \right)$$

here \tilde{k} – plane bulk modulus, self-strain tensor has a form $\boldsymbol{\varepsilon}^p = \frac{\vartheta^p}{2} \mathbf{E}$. Then

$$\begin{aligned}
\mathbf{q}_- &= \left(\mu_1^- - \tilde{k}_1^- \right) \text{tr}(\boldsymbol{\varepsilon}) \mathbf{E} - 2\mu_1^- \boldsymbol{\varepsilon}, & \mathbf{q}_+ &= \left(\left(\mu_1^+ - \tilde{k}_1^+ \right) \text{tr}(\boldsymbol{\varepsilon}) + \tilde{k}_+ \vartheta^p \right) \mathbf{E} - 2\mu_1^+ \boldsymbol{\varepsilon} \\
\gamma_*^+ &= \gamma - \frac{1}{2} \frac{\tilde{k}_0 \tilde{k}_+}{\tilde{k}_1^+} (\vartheta^p)^2, & \mu_1^{\pm} &= \mu_{\pm} - \mu_0, & \tilde{k}_1^{\pm} &= \tilde{k}_{\pm} - \tilde{k}_0
\end{aligned} \tag{8}$$

and (3) may be written:

$$\begin{aligned}
\Pi_1 - \Pi_0 &= \frac{1}{2} V \left(\left(\tilde{k}_1^- - \mu_1^- \right) \text{tr}(\boldsymbol{\varepsilon}_-) \text{tr}(\boldsymbol{\varepsilon}_0) + 2\mu_1^- \boldsymbol{\varepsilon}_- : \boldsymbol{\varepsilon}_0 \right) \\
\Pi_+ - \Pi_0 &= \frac{1}{2} V \left(2\gamma_*^+ + \frac{\left(\tilde{k}_+ \vartheta^p \right)^2 - \tilde{k}_1^+ \tilde{k}_+ \vartheta^p \left(\text{tr}(\boldsymbol{\varepsilon}_+) + \text{tr}(\boldsymbol{\varepsilon}_0) \right)}{\tilde{k}_1^+} + \right. \\
&\quad \left. \left(\tilde{k}_1^+ - \mu_1^+ \right) \text{tr}(\boldsymbol{\varepsilon}_+) \text{tr}(\boldsymbol{\varepsilon}_0) + 2\mu_1^+ \boldsymbol{\varepsilon}_+ : \boldsymbol{\varepsilon}_0 \right)
\end{aligned} \tag{9}$$

It should be noted that the equation (1) determines not two but only one material with the volume free energy density represented at Fig.1 by a bold line. Everyone can see that both phases can exist not under every external strain field. So our first task is construction of domains where one-phase states exist. We should construct a curve in strain-space for which $f_+(\boldsymbol{\varepsilon}^+(\boldsymbol{\varepsilon}_0)) = f_-(\boldsymbol{\varepsilon}^+(\boldsymbol{\varepsilon}_0))$ and $f_+(\boldsymbol{\varepsilon}^-(\boldsymbol{\varepsilon}_0)) = f_-(\boldsymbol{\varepsilon}^-(\boldsymbol{\varepsilon}_0))$. If $f_+(\boldsymbol{\varepsilon}^+(\boldsymbol{\varepsilon}_0)) \leq f_-(\boldsymbol{\varepsilon}^+(\boldsymbol{\varepsilon}_0))$ then the inclusion can be only in the plus state. The deformations for which $f_-(\boldsymbol{\varepsilon}^-(\boldsymbol{\varepsilon}_0)) \leq f_+(\boldsymbol{\varepsilon}^-(\boldsymbol{\varepsilon}_0))$ is true correspond “-” state.

Typical domains of one-phase states and the switch surface are represented on the Fig.2. The phase “-” can exist only inside the gray area, while the hatched area corresponds to the “+” state. The dotted line denotes the hydrostatic deformation. One can see from Fig.2. that under some deformations both states can exist.

The thick line denotes the “switch” surface. It corresponds to the condition $\Pi_- = \Pi_+$. It determines when the first one-phase state of an inclusion is changed by another one.

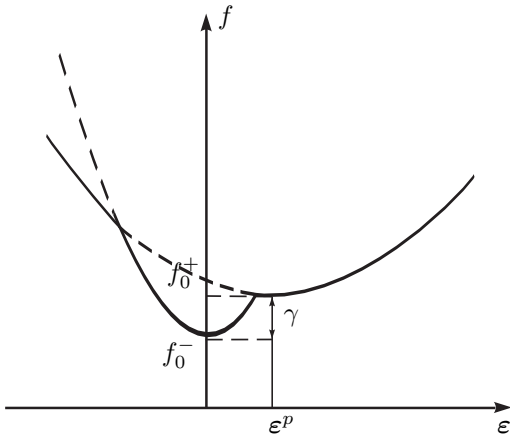


Figure 1: Free energy.

For example, at Fig.2(a) the inclusion can be only in “-” phase in the undeformed state. When we reach point E the “+” state becomes possible but its potential energy is greater than for “-” one. After crossing the switch surface phase “+” became more preferable from the energy point of view and if loading path goes out of the gray area, it means then the inclusion could not be in the “-” state any more.

The transformation surface can be closed or unclosed depending on the material parameters. (Fig.2 (a),(b)). It means that on some loading paths phase transitions are not allowed whatever the values of strains are.

It should be noted that the location of the domains and the switch surface also depends on the material parameters. The curves can be replaced from one position to another with γ changing. As a result phase transformation can happen without any external stress field only because of the temperature changes.

2 Two-phase states of the inclusion under all-rounding stretching

In the previous section it was assumed that the inclusion can be only in one of the one-phase states. Let us consider the possibility of two-phase states. A problem of equilibrium two-phase configurations of elastic bodies can be reduced to the determination of the interface Γ and corresponding displacement field \mathbf{u} that is smooth enough at material points $\mathbf{x} \notin \Gamma$ and continuous on Γ

$$\mathbf{x} \in \Gamma, \quad [\mathbf{u}] = 0, \quad (10)$$

and satisfies boundary condition, the equilibrium conditions in a volume

$$\mathbf{x} \notin \Gamma, \quad \nabla \cdot \boldsymbol{\sigma} = 0, \quad \theta = \text{const}, \quad (11)$$

and conditions on the interface:

$$[\boldsymbol{\sigma}] \cdot \mathbf{n} = 0, \quad (12)$$

$$[f] - \boldsymbol{\sigma} : [\boldsymbol{\varepsilon}] = 0 \quad (13)$$

here \mathbf{n} — is the normal to the interface Γ .

The traction continuity conditions (12) follows from equilibrium considerations. An additional thermodynamic condition (13) (the Maxwell relation) arise from the additional degree of freedom produced by a free phase boundary. This condition acts as the restriction on the geometry of the interface. As a problem with an unknown boundary, the problem of equilibrium two-phase configurations can be solved using semi-inverse method, when the shape of the interface is presupposed and the geometrical parameters are determined by the requirement of the thermodynamic condition. As a rule, the solution, if exist, is non-unique. Various two-phase structures can satisfy equilibrium conditions at the same boundary conditions. Some of them are unstable. The choice of the solution can be made then on the base of the analysis of stability and estimations of energy changes due to phase transformations.

At first let us study the possibility of equilibrium cylindrical symmetric two-phase deformations with several cylindrical interfaces of radius ρ_i which divide the body into the regions

$$V_i = \{\varphi \in [0, 2\pi], \rho_{i-1} < r < \rho_i, \quad (i = 1, 2, \dots, N + 1)\} \quad (14)$$

where φ, r — are the polar coordinates, $\rho_{N+1} = 1$.

Radial displacement are given by

$$u_i = A_i r + D_i / r, \quad (i = 1, \dots, N + 1) \quad (15)$$

where A_i, D_i are determined by the conditions:

$$[u]_{\rho_i} = 0, \quad [\sigma_r]_{\rho_i} = 0, \quad \varepsilon_r|_{r \rightarrow \infty} = \varepsilon^0, \quad u|_{r=0} < \infty \quad (16)$$

Suppose that $N \geq 2$ and show that the assumption lead to the contradictions. Rewrite the thermodynamic condition as:

$$2\gamma_* + \frac{d_+}{d_- \tilde{k}_1} \left(\tilde{k}_+ \vartheta^p - 2\tilde{k}_i A_i \right)^2 + \frac{4\mu_1 d_-^* B_i^2}{d_- \rho^4} + \frac{4\mu_1 \left(\tilde{k}_+ \vartheta^p - 2\tilde{k}_i A_i \right) B_i}{d_- \rho^2} = 0 \quad (17)$$

$$\gamma_* = \gamma - \frac{\tilde{k}_+ \tilde{k}_- (\vartheta^p)^2}{2k_1},$$

$$k_1 = \tilde{k}_+ - \tilde{k}_-, \quad \mu_1 = \mu_+ - \mu_-, \quad d_{\pm} = \tilde{k}_{\pm} + \mu_-, \quad d_-^* = \tilde{k}_- + \mu_+$$

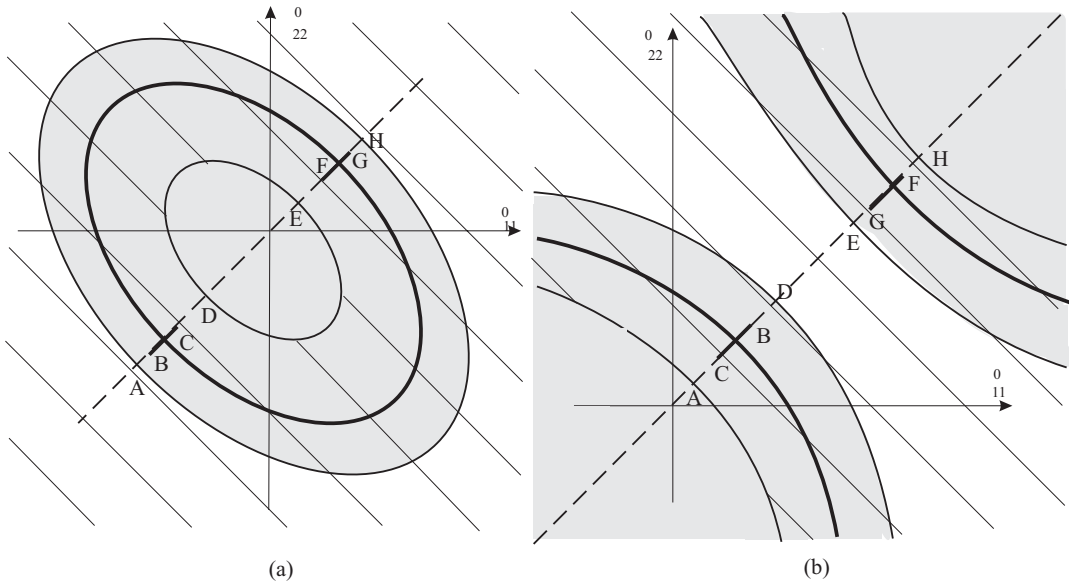


Figure 2: Phase domains and switch surfaces. (a) $\tilde{k}_+ < \tilde{k}_-, \mu_+ < \mu_-$, (b) $\tilde{k}_+ > \tilde{k}_-, \mu_+ < \mu_-$

Without loss of generality suppose that V_1 is occupied by the phase “-”, V_2 — phase “+”, etc.
Components of the displacement in the polar coordinates are:

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\varphi = \frac{u}{r} \quad (18)$$

it follows from (2), (15) and (16) that the coefficients A_i, D_i have to satisfied:

$$\begin{aligned} A_1 \rho_1 &= A_2 \rho_1 + D_2 / \rho_1 \\ 2\tilde{k}_- A_1 &= 2\tilde{k}_+ A_2 - \tilde{k}_+ \vartheta^p - 2\mu_+ D_2 / \rho_1^2 \\ A_2 \rho_2 + D_2 / \rho_2 &= A_3 \rho_2 + D_3 / \rho_2 \\ 2\tilde{k}_+ A_2 - \tilde{k}_+ \vartheta^p - 2\mu_+ D_2 / \rho_2^2 &= 2\tilde{k}_- A_3 - 2\mu_- D_3 / \rho_2^2 \\ A_3 + D_3 &= \varepsilon_0 + D_0 \\ 2\tilde{k}_- A_3 - 2\mu_- D_3 &= 2\tilde{k}_0 \varepsilon_0 - 2\mu_0 D_0 \end{aligned}$$

From the above system follows that

$$A_3 - A_1 = \frac{\mu_1 D_3}{d_+ \rho_2^2} \quad (19)$$

Substituting this dependence into (17) on the second interface leads to

$$2\gamma_* + \frac{d_+}{d_- k_1} \left(\tilde{k}_+ \vartheta^p - 2\tilde{k}_1 A_1 \right)^2 - \frac{4\mu_1^2 k_1 D_3^2}{d_- d_+ \rho_2^4} = 0,$$

but the equation $2\gamma_* + \frac{d_+}{d_- k_1} \left(\tilde{k}_+ \vartheta^p - 2\tilde{k}_1 A_1 \right)^2 = 0$ has to be hold. It follows that $D_3 / \rho_2^2 = 0$. The last leads to $D_3 = 0, A_1 = A_3$. So, no more than one interface is allowed.

Further we study a possibility of equilibrium cylindrical symmetric two-phase deformations. Suppose that a cylindrical interface $0 \leq \rho_\Gamma \leq 1$ exists and divides the inclusion into regions $V_i : r < \rho_\Gamma$ and $V_e = V \setminus V_i$. The new phase area can appear and spread during the loading either from the center of the inclusion as a cylindrical nucleus or from the surface in the form of a cylindrical

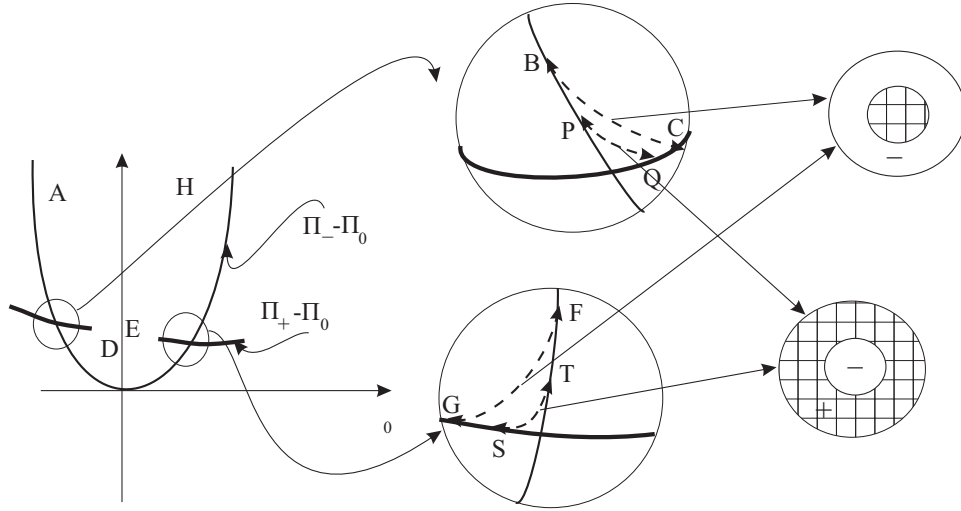


Figure 3: The dependence of the potential energies on the external field. (1) $\tilde{k}_+ > \tilde{k}_-, \mu_+ > \mu_-$, (2) $\tilde{k}_+ < \tilde{k}_-, \mu_+ < \mu_-$ ($\mu_{e,i} > \mu_0$)

layer. We look for the radial displacements in the matrix and V_i, V_e of the form (15). From the equilibrium equations it follows:

$$A_i = \frac{2\varepsilon_0 (\tilde{k}_0 + \mu_0) d_e + d_e \tilde{k}_i \vartheta_i + t (1 - \rho_\Gamma^2)}{2 \left((\tilde{k}_e + \mu_0) d_i - (\mu_0 - \mu_e) (\tilde{k}_i - \tilde{k}_e) \rho_\Gamma^2 \right)},$$

$$A_e = \frac{2\varepsilon_0 (\tilde{k}_0 + \mu_0) d_i + d_i \tilde{k}_e \vartheta_e + t \rho_\Gamma^2}{2 \left((\tilde{k}_e + \mu_0) d_i - (\mu_0 - \mu_e) (\tilde{k}_i - \tilde{k}_e) \rho_\Gamma^2 \right)}, \quad (20)$$

$$D_i = 0, \quad A_0 = \varepsilon_0, \quad d_{i,e} = \tilde{k}_{i,e} + \mu_e, \quad t = (\mu_0 - \mu_e) (\tilde{k}_i \vartheta_i - \tilde{k}_e \vartheta_e)$$

The equilibrium radius can be found with the help of the thermodynamic equation. In view of (18), (20) $\varepsilon_r^i = \varepsilon_\varphi^i = A_i$. It follows that $\mathbf{q}^i = -\mathbf{C}_1 : \varepsilon^i + \mathbf{C}_+ : \varepsilon^p$ ($\mathbf{C}_1 = \mathbf{C}_+ - \mathbf{C}_-$) is spherical on V_i

$$\mathbf{q}^i = q \mathbf{E}, \quad q = -2k_1 A_i + \tilde{k}_+ \vartheta^p, \quad (21)$$

On the other hand q is determined by the Maxwell relation, i.e. entirely by material properties:

$$q^2 = -2\gamma_* \frac{d_e}{d_i} k_1 \quad (22)$$

And (22) because of (20), (21), yields

$$\rho_\Gamma^2 = \pm \left(\frac{(\tilde{k}_e + \mu_0) d_i}{k_1 (\mu_0 - \mu_e)} + \frac{d_e}{q_* (\mu_0 - \mu_e)} \left(2 (\tilde{k}_0 + \mu_0) \varepsilon_0 - \vartheta_* \right) \right) \quad (23)$$

$$\vartheta_* = \frac{(\tilde{k}_e + \mu_0) \tilde{k}_+ \vartheta^p}{k_1}$$

where q_* is defined by (22). “+” agrees with the advent of the new phase in the center of the inclusion, “-” — from the surface.

To begin with, assume $V_i = V_+$. Letting $\rho_\Gamma = 0$ and $\rho_\Gamma = 1$, one can find deformations ε_{0+} and ε_{1+} . ε_{0+} corresponds to the appearance of the new phase in the center while ε_{1+} — the completion of the transformation. Analogously, ε_{1-} and ε_{0-} can be obtained for the case of $V_i = V_-$. Two-phase states of the inclusion exist on the lines BC and FG (Figure2).

2.1 Energy changes due to phase transformation

To choose the solution we analyze energy changes. Estimation of the energy of the two-phase state Π_{12} can be made as follows

$$\Pi_{12} - \Pi_0 = \Pi_{12} - \Pi_e + \Pi_e - \Pi_0, \quad (24)$$

here Π_e — energy of the medium with the inclusion in the state e . $\Pi_e - \Pi_0$ defined by (3). So (24) may be written as:

$$\begin{aligned} \Pi_{12} - \Pi_e &= \int_{V_0} (f_0(\boldsymbol{\varepsilon}_{12}) - f_0(\boldsymbol{\varepsilon}_e^0)) dV + \int_{V_e} (f_e(\boldsymbol{\varepsilon}_{12}) - f_e(\boldsymbol{\varepsilon}_e)) dV \\ &\quad + \int_{V_i} (f_i(\boldsymbol{\varepsilon}_{12}) - f_e(\boldsymbol{\varepsilon}_e)) dV \end{aligned}$$

The relation (1) leads to

$$\begin{aligned} f_0(\boldsymbol{\varepsilon}_{12}) - f_0(\boldsymbol{\varepsilon}_e^0) &= \frac{1}{2} (\boldsymbol{\sigma}_{12} + \boldsymbol{\sigma}_e^0) : (\boldsymbol{\varepsilon}_{12} - \boldsymbol{\varepsilon}_e^0) \\ f_e(\boldsymbol{\varepsilon}_{12}) - f_e(\boldsymbol{\varepsilon}_e) &= \frac{1}{2} (\boldsymbol{\sigma}_{12} + \boldsymbol{\sigma}_e) : (\boldsymbol{\varepsilon}_{12} - \boldsymbol{\varepsilon}_e) \\ f_i(\boldsymbol{\varepsilon}_{12}) - f_e(\boldsymbol{\varepsilon}_e) &= \pm \left(\gamma_* + \frac{1}{2} \mathbf{q}_{12} : \mathbf{C}_1^{-1} : \mathbf{q}_e^0 + \frac{1}{2} (\boldsymbol{\sigma}_{12} + \boldsymbol{\sigma}_e^0) : (\boldsymbol{\varepsilon}_{12} - \boldsymbol{\varepsilon}_e^0) \right) \\ \mathbf{q}_{12} &= \left(-2\tilde{k}_1 A_i + \tilde{k}_+ \vartheta^p \right) \mathbf{E}, \quad \mathbf{q}_e^0 = \left(-2\tilde{k}_1 A_e(0) + \tilde{k}_+ \vartheta^p \right) \mathbf{E}, \quad \gamma_* = \gamma - \frac{1}{2} \frac{\tilde{k}_- \tilde{k}_+}{\tilde{k}_1} (\vartheta^p)^2 \end{aligned}$$

“+” assumes that the new phase spreads from the center of the inclusion, “-” — from its surface, $A_e(0) = A_e|_{\rho_\Gamma=0}$. One finds that:

$$\Pi_{12} - \Pi_e = \pm \int_{V_i} \left(\gamma_* + \frac{1}{2} \mathbf{q}_{12} : \mathbf{C}_1^{-1} : \mathbf{q}_e^0 \right) dV$$

The dependence of the potential energies on the external field are shown in the Figure.3 ($\mu_{i,e} > \mu_0$). The thick lines denote $\Pi_+ - \Pi_0$, the dotted lines — $\Pi_{12} - \Pi_0$. The energy of the two-phase states at the beginning of phase transformations represented by points B, P, F, T . The points C, Q, G, S correspond to complete transformation of the inclusion into plus state. The comparison of the energies of the two-phase configurations has shown that the energy of the two-phase inclusion is less if the phase with the greater modulus is placed inside the interface. The competition between two-phase and one-phase states determines if the transition from one phase to another will be smooth because of the stable equilibrium phase interface existence, or one-phase state will be changed abruptly by another one in the inclusion. It is seen that in this case the energies of the one-phase states are more preferable than the both two-phase states at the same boundary conditions in displacement. Energy preference of two-phase states depends on relation between shear modulus of the phases and the matrix. In the case when $\mu_0 > \mu_+$ the two-phase configuration with the cylindrical nucleus become more preferable from the energy point of view than the one-phase state. If $\mu_0 > \mu_-$ the appearance of new phase from the surface of the inclusion decreasing the energy of the system.

For analysis of stability let $\partial^2 F / \partial z^2 = \partial^2 (\Pi_{12} - \Pi_0) / \partial z^2$, where $z = \rho_\Gamma^2$. Differentiating (2.1) twice ends up in:

$$\frac{\partial^2 F}{\partial z^2} = \frac{2(\mu_e - \mu_0)\gamma_* k_1}{d_i (\tilde{k}_e + \mu_0) + z(\mu_e - \mu_0)\tilde{k}_1}. \quad (25)$$

From the fact that $0 \leq z \leq 1$, one can conclude that the denominator is always positive. Keeping in mind that $\gamma_* k_1 < 0$ as follows from (22) one confirms that the sign of (25) depends on the relation between shear modulus of the matrix and the phase placed outside of the interface.

It should be noted that the stability or metastability of the solutions depends on the type of the boundary conditions. This fact was pointed to [4],[1],[5],[6] where the problem of the phase transformation of the isotropic sphere has been studied. It was shown that two-phase configurations are stable if radial displacements on the external surface of a sphere are given, and unstable if pressure is given. In our case we rather have second problem. Let us study how the situation will be changed for an inclusion in a finite body.

Consider a cylindrical inclusion in the cylindrical body with a radius R . Displacements u_0 are given on the external surface of the body. The procedure described above leads to the following dependence of the equilibrium interface radius on the external parameters:

$$\begin{aligned} \rho_\Gamma^2 &= \pm \rho^2 \frac{(\rho^2 \tilde{k}_0 + (1 - \rho^2) \tilde{k}_e + \mu_0) d_i + d_e (k_1 (\tilde{k}_0 + \mu_0) \vartheta_0 - \vartheta_*) / q_*}{(\rho^2 (\tilde{k}_0 + \mu_e) + \mu_0 - \mu_e) k_1} \\ \vartheta_* &= (\tilde{k}_- + \mu_0 + \rho^2 (\tilde{k}_0 - \tilde{k}_-)) \tilde{k}_+ \vartheta^p, \quad \rho = r_1 / R, \quad \rho_\Gamma = r_\Gamma / R \end{aligned}$$

here $\vartheta_0 = 2u_0 / R$, r_1 , r_Γ — radiuses of the inclusion and the interface. One can see that if $\rho \rightarrow 0$ expression for ρ_Γ^2 / ρ^2 is the same as for the unbounded medium(23). Difference of the potential energies of the two-phase and one-phase states can be written:

$$\frac{1}{R^2} (\Pi_{12} - \Pi_e) = \pm \rho_\Gamma^2 \left(\gamma_* + \frac{1}{2} \mathbf{q}_{12} : \mathbf{C}_1^{-1} : \mathbf{q}_e^0 \right)$$

It leads that on equilibrium interface

$$\frac{1}{R^2} \frac{\partial^2 F}{\partial z^2} = \frac{2(\mu_e - \mu_0 - \rho^2 (\tilde{k}_0 + \mu_e)) \gamma_* k_1}{\rho^2 \left(d_i (\tilde{k}_e (1 - \rho^2) + \mu_0 + \rho^2 \tilde{k}_0) + \rho_\Gamma^2 (\mu_e - \mu_0 - \rho^2 (\tilde{k}_0 + \mu_e)) \right)} \quad (26)$$

Because $\rho_1^2 \leq \rho^2$, the denominator of (26) is positive. Therefore the stability of two-phase configurations depends on a sign of $\mu_e - \mu_0 - \rho^2 (\tilde{k}_0 + \mu_e)$. Thus in the case of unbounded body, the stability of two-phase states depends on the relation between shear modulus of the phases and the matrix but in addition to that size effect takes place: stability of the interface is determined by the relative dimensions of the inclusion and the body. If $\mu_e > \mu_0$ then increase of the radius of the inclusion leads to the instability of the two-phase state. If $\mu_0 > \mu_e$ then there is a critical relative dimension $\rho = (\mu_e - \mu_0)/(\tilde{k}_0 + \mu_e)$ from which the two-phase state is more preferential than the one phase state.

2.2 Stability of the cylindrical symmetric equilibrium states

In this section we study the infinitesimal stability of two-phase cylindrical symmetric deformations. In the stability examination of the obtained solutions we follow the approach developed by Eremeev and Zubov [7]. We introduce the vector w of the displacement perturbations and the perturbation of the interface η . Let us examine the stability of the solutions with respect to the perturbations:

$$\mathbf{u} = \mathbf{u}^0(r) + \mathbf{w}(r, \varphi), \quad r = r^0 + \eta(\varphi) \quad (27)$$

$$\mathbf{w}(r, \varphi) = u(r, \varphi)\mathbf{e}_r + v(r, \varphi)\mathbf{e}_\varphi \quad (28)$$

here $\mathbf{u}^0(r) = u^0(r)\mathbf{e}_r$ the equilibrium solution, $\mathbf{e}_r, \mathbf{e}_\varphi$ — are the unit vectors associated with the polar coordinates.

Linearization of the equilibrium equations leads to

$$(\tilde{k} + \mu) \frac{\partial \psi}{\partial r} - \frac{2\mu}{r} \frac{\partial \omega}{\partial \varphi} = 0 \quad (29)$$

$$(\tilde{k} + \mu) \frac{1}{r} \frac{\partial \psi}{\partial \varphi} + 2\mu \frac{\partial \omega}{\partial r} = 0$$

$$\psi \triangleq \nabla \cdot \mathbf{w} = \frac{\partial u}{\partial r} + \frac{1}{r} \left(u + \frac{\partial v}{\partial \varphi} \right), \quad \omega = \frac{\partial v}{\partial r} + \frac{1}{r} \left(v - \frac{\partial u}{\partial \varphi} \right). \quad (30)$$

The kinematic and static conditions on the interface can be written

$$[u] = -\eta \left[\frac{du^0}{dr} \right], \quad [v] = 0, \quad (31)$$

$$\left[(\tilde{k} - \mu) \psi + 2\mu \frac{\partial u}{\partial r} \right] = -\eta \left[\frac{d\sigma_r^0}{dr} \right], \quad \left[2\mu \frac{\partial v}{\partial r} \right] = \frac{d\eta}{d\varphi} [\sigma_\varphi]$$

The boundary conditions correspond to $\mathbf{w} = 0$ and the conditions on the boundary of the matrix and the inclusion have the appearance of (31) when $\eta = 0$.

Following [7], we are looking for the solution (27) as a series:

$$u(r, \varphi) = \sum_{n=2}^{\infty} U_n(r) \cos(n\varphi), \quad v(r, \varphi) = \sum_{n=2}^{\infty} V_n(r) \sin(n\varphi), \quad (32)$$

$$\eta(\varphi) = \sum_{n=2}^{\infty} \zeta_n(r) \cos(n\varphi)$$

Upon substitution (32) into (27) one arrives at the system of differential equations:

$$(\tilde{k} + \mu) r \Psi'_n(r) - 2\mu n \Omega_n(r) = 0$$

$$(\tilde{k} + \mu) n \Psi_n(r) + 2\mu r \Omega_n(r) = 0 \quad (33)$$

$$\Psi_n(r) = U'_n(r) + \frac{1}{r} (U_n(r) + nV_n(r)), \quad \Omega_n(r) = V'_n(r) + \frac{1}{r} (V_n(r) + nU_n(r))$$

From which the coefficients may be founded:

$$U_n(r) = \frac{(\mu - \tilde{k})n + 4\mu}{8(n+1)\mu} r^{n+1} C_n^1 + \frac{(\mu - \tilde{k})n - 4\mu}{8(n-1)\mu} r^{1-n} C_n^2 + r^{n-1} D_n^1 + r^{-n-1} D_n^2 \quad (34)$$

$$V_n(r) = -\frac{(\mu - \tilde{k})n - 2(\tilde{k} + \mu)}{8(n+1)\mu} r^{n+1} C_n^1 + \frac{(\mu - \tilde{k})n + 2(\tilde{k} + \mu)}{8(n-1)\mu} r^{1-n} C_n^2 - r^{n-1} D_n^1 + r^{-n-1} D_n^2$$

Linearisation of the thermodynamic equation on the interface is equivalent to:

$$\frac{q}{d_e} \left((\tilde{k}_i + \mu_i) U'_n + \frac{1}{\rho_\Gamma} (\tilde{k}_i - \mu_i + 2\mu_e) (nV_n + U_n) \right) = -\frac{1}{k} \frac{d\zeta_n}{dt} \quad (35)$$

and describes the further evolution of initial perturbation of the interface.

Then (34) and (35), after lengthy computation, lead to η

$$\frac{1}{k} \frac{d\zeta_n}{dt} = L_n(\vartheta_0) \zeta_n \quad (36)$$

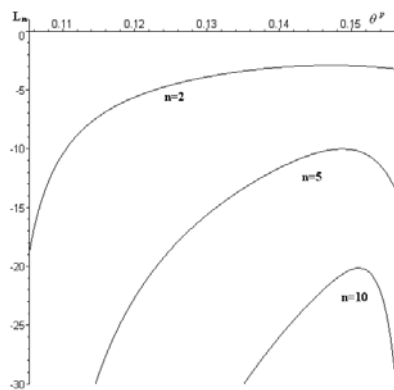


Figure 4: Dependence L_n on external field. $\tilde{k}^+ > \tilde{k}^-$, $\mu^+ > \mu^- > \mu_0$

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Elena N. Vilchevskaya, Alexander B.Freidin, IPME, Bolshoi 61, V.O., St.Petersburg, Russia

The roots ϑ_* of the equation $L_n(\vartheta_0) = 0$ are the points of bifurcation of the boundary value problem. The behavior of small perturbations near a point of bifurcation depends on a sign of coefficient L_n of (36) Negative values of L_n correspond to exponential decrease of amplitude of the interface perturbation with respect to time. Positive values of L_n correspond to the growth of perturbation. Then the number n characterizes the mode of loss of stability. It was shown that if $\mu_e - \mu_0 - \rho^2 (\tilde{k}_0 + \mu_e) > 0$ is true, then two-phase state is stable to the radial perturbation.

From the results, presented in Fig.4 it follows that two-phase deformation of the inclusion with a phase with a greater shear modulus placed in the central part of the cylinder does not have any bifurcation points for external fields that makes two-phase states possible.