Characterization and stability of two-phase piecewise-homogeneous deformations

By Y.B. Fu[1] and A. B. Freidin[2]

[1]*Department of Mathematics, Keele University, Staffordshire ST5 5BG, UK*
[2]*Institute of Mechanical Engineering Problems, Russian Academy of Sciences Bolshoi pr. 61, V.O., St. Petersburg 199178, Russia*

Many solid materials exhibit stress-induced phase transformations. Such phenomena can be modelled with the aid of the nonlinear elasticity theory with appropriate choices of the strain-energy function. It was shown by Gurtin (1983) that if a two-phase deformation (with gradient $\mathbf{F}$) in a finite elastic body is a local energy minimizer, then given any point $\mathbf{p}$ of the surface of discontinuity, the piecewise-homogeneous deformation corresponding to the two values $\mathbf{F}^{\pm}(\mathbf{p})$ of $\mathbf{F}(\mathbf{p})$ is a global energy minimizer. Thus, instability of the latter state would imply instability of the former state. In this paper we investigate the stability properties of such piecewise-homogeneous deformations. More precisely, we are concerned with two joined half-spaces that correspond to two different phases of the same material. We first show how such a two-phase deformation can be constructed. Then the stability of the piecewise-homogeneous deformation is investigated with the aid of two test criteria. One is a kinetic stability criterion based on a quasi-static approach and on the growth/decay behaviour of the interface in the undeformed configuration when it is perturbed; the other, referred to as the energy criterion, is by determining whether the deformation is a minimizer of the total energy with respect to perturbations of the interface in both the current and undeformed configurations. We clarify the differences between the two criteria, and provide a compact formula which can be used to establish the stability/instability of any two-phase piecewise-homogeneous deformations.

Keywords: phase transformation, stability, interfacial instability

1. Introduction

It is well-known that stress-induced phase transformations can be modelled with the aid of the nonlinear elasticity theory with appropriate choices of the strain-energy function. There are three major issues that need to be resolved. These are (i) characterization of materials that can support multi-phase deformations (the question of existence), (ii) description of multi-phase deformations under various loading conditions when the strain-energy function and geometry of the elastic body are specified, and (iii) determination of the stability of multi-phase deformations that can exist mathematically. Many previous studies have been devoted to the resolution of the first issue. For instance, existence of multi-phase deformations was investigated by Knowles and Sternburg (1978), Ball (1980), James (1981), Rosakis (1990), Rosakis and Jiang (1993) for compressible materials, and by Abeyaratne (1980), Abeyaratne and Knowles (1989) for incompressible materials. All these studies except that of Ball (1980) were concerned with isotropic materials. One important result that emerged from such studies is that a necessary condition for the existence of multi-phase deformations is that the material loses strong ellipticity at some deformation gradient. Studies in the same spirit for transversely isotropic materials have been carried out by Qiu and Pence (1997), Merodio and Pence (2001a, b), and Merodio and Ogden (2002, 2003). In a recent series of papers, Freidin and Chiskis (1994a, b), Morozov and Freidin (1998), Freidin and Croitoro (1999), and Freidin *et al.* (2002), it was demonstrated that more explicit results for 3-D deformations could be obtained by constructing so-called *phase transition zones* (PTZs). For isotropic materials, a PTZ is a subspace of $\mathbf{R}^3$ to which the three principal invariants of the Cauchy-Green deformation tensors (or equivalently the three principal stretches) on either side of a phase boundary must necessarily belong. In characterizing the PTZ for a generally isotropic material, the existence conditions for multi-phase states were most naturally obtained.

Detailed descriptions of multi-phase deformations for non-convex strain-energy functions seem to have been pioneered by Ericksen (1975) who considered the simple problem of a stretched elastic bar and illustrated the important role played by the one-dimensional Maxwell relation in the stability considerations. Following Ericken's (1975) approach, Abeyaratne (1981) considered the finite twisting of an incompressible elastic tube, Fosdick and MacSithigh (1983) studied the helical shear of a compressible elastic tube, Fosdick and Zhang (1993, 1994, 1995a, b) considered a variety of torsion and anti-plane shear problems involving elastic cylindrical tubes, and more recently, Tommasi *et al.* (2001) considered an incompressible elastic body in a state of homogeneous plane strain. It was demonstrated in these studies that depending on the loading conditions the energy minimizer may consist of a single phase or a mixture of two phases.

The present paper addresses the third issue listed above. We consider the stability of an infinite space of elastic material that has divided into two half-spaces through a stress-induced phase transformation, the deformation in each half-space being homogeneous. This is an idealized model problem, but its importance can be seen from the paper by Gurtin (1983) where the author showed that if a two-phase deformation (with gradient $\mathbf{F}$) is a local energy minimizer, then given any point $\mathbf{p}$ of the surface of discontinuity, the piecewise-homogeneous deformation corresponding to the two values $\mathbf{F}^{\pm}(\mathbf{p})$ of $\mathbf{F}(\mathbf{p})$ is a global energy minimizer. Thus, instability of the latter state would imply instability of the former state. The problem examined in the present paper can be viewed as a building block in determining whether a multi-phase inhomogeneous deformation is a stable deformation or not.

Our present study is in the same spirit as those of Eremeyev and Zubov (1991), Eremeyev (1999) and Eremeyev *et al.* (2003) in the sense that in assessing the stability of the two-phase deformation, we allow the phase boundary in the reference configuration to be variable and to take part in the energy minimization. Thus, the present study differs from that of Simpson and Spector (1991) in which the interface in the reference configuration between the two half-spaces was not allowed to vary and hence the problem addressed by them was essentially a joint-body problem.

The rest of this paper is organized as follows. In the next section, we present the general governing equations for stress-induced phase transformations of elastic bodies. An important issue in the studies of phase transformations is how to characterize a two-phase state and conditions for its existence. In Section 3 we explain a procedure through a simple example and relate it to a general procedure that has been developed by Freidin and Chiskis (1994a, b). In Section 4, we derive the incremental governing equations when the primary two-phase state discussed in Section 3 is subjected to general perturbations. The perturbations include a perturbation of the position of the interface in the undeformed configuration and perturbations of displacement in the current configuration. We use a quasi-static approach in which only the position of the interface in the undeformed configuration is assumed to depend on time. When this time dependence is also neglected, the governing equations are those for an adjacent equilibrium state. Following Eremeyev *et al.* (2003), we write down a kinetic stability criterion based upon the observation that the rate of energy dissipated as the interface traverses the material in the undeformed configuration should be non-negative. In Section 5, we consider perturbations in the form of a normal mode which is sinusoidal in the direction parallel to the unperturbed interface and decays exponentially along both directions orthogonal to the unperturbed interface. Our analysis draws upon recent results of Fu and Mielke (2002) and Mielke and Fu (2004) for the half-space problem. As a result, the bifurcation condition and the condition for kinetic stability are written in an explicit form that is very amenable to computations even in the general case. In Section 6, we derive a similar explicit expression for the energy increment when the primary two-phase deformation is subjected to normal-mode perturbations. In Section 7, a numerical example is used to illustrate the fact that when perturbations/variations of the interface in the undeformed configuration are also allowed, the region of stability is only a subset of the corresponding region of stability when such perturbations/variations of the interface are not allowed. In the final section, we summarize our results and make some additional comments.

2. Governing equations

Let the static deformation of an elastic body be given by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{2.1}$$

which assigns position $\mathbf{x}$ to the material point that occupies position $\mathbf{X}$ in the undeformed (reference) configuration. In this paper we shall study deformations associated with stress-induced phase transformations. At an interface between two different phases of the same material, at least one of the first derivatives of $x_i(X_A)$ suffers a discontinuity, where x_i and X_A are the coordinates of $\mathbf{x}$ and $\mathbf{X}$ relative to a common rectangular coordinate system. We shall follow the convention that lower case subscripts are associated with the coordinates of $\mathbf{x}$ and upper case subscripts with the coordinates of $\mathbf{X}$. The jump of a function f across a phase interface is defined by

$$[f] = f^+ - f^-, \tag{2.2}$$

where superscripts "+" and "−" signify evaluation at the interface as it is approached from the two sides respectively. To avoid using double superscripts, we shall replace a superscript "+" or "−" by the corresponding subscript on quantities that have already another superscript. Thus, for instance, f^+ and g_+^2 both signify evaluation on the "+" side of an interface. When there is no "+" or "−" superscript/subscript attached to a field variable evaluated at the interface, it means that the variable can be evaluated on either side of the interface.

The behaviour of an elastic body is completely described by its strain-energy function W which is taken to be a C^2 function of the deformation gradient $\mathbf{F}$. Thus, we write $W = W(\mathbf{F})$, where

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad F_{iA} = x_{i,A}, \tag{2.3}$$

and a comma signifies partial differentiation. The first Piola-Kirchhoff stress tensor $\boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi}^T = \frac{\partial W}{\partial \mathbf{F}}, \quad \pi_{iA} = \frac{\partial W}{\partial F_{iA}}. \tag{2.4}$$

The equilibrium equation is given by

$$\mathrm{Div}\boldsymbol{\pi}^T = \mathbf{0}, \quad \pi_{iA,A} = 0, \tag{2.5}$$

and its weak form is

$$[\boldsymbol{\pi}\mathbf{N}] = \mathbf{0}, \quad [\pi_{iA}N_A] = 0, \tag{2.6}$$

where $\mathbf{N}$ denotes the unit vector normal to the interface in the reference configuration and points from the "+" phase into the "−" phase. The jump condition (2.6) expresses continuity of traction across the interface.

The conservation law (2.5) may also be written as

$$\mathrm{Div}\boldsymbol{M} = \mathbf{0}, \quad M_{AB,A} = 0, \tag{2.7}$$

with corresponding weak form

$$[\mathbf{M}^T\mathbf{N}] = \mathbf{0}, \quad [M_{AB}N_A] = 0, \tag{2.8}$$

where $\mathbf{M}$, defined by

$$\boldsymbol{M} = W\mathbf{I} - \boldsymbol{\pi}^T\mathbf{F},$$

is the Eshelby energy-momentum tensor (see Eshelby (1975)). Noether's (1918) theorem states that for a system arising from a variational principle, every symmetry of the variational principle gives rise to a conservation law. The conservation laws (2.5) and (2.7) arise from the invariance of the energy functional with respect to translations in $\mathbf{x}$ and $\mathbf{X}$, respectively (Olver 1986, p. 281). As another interpretation, equations (2.5) and (2.7) are necessary conditions for the deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$ to be a stationary point of the energy functional when variations in $\mathbf{x}$ and $\mathbf{X}$ are considered, respectively. Correspondingly, the jump conditions are necessary conditions for the energy functional to be stationary when the interface is perturbed in the current and reference configurations, respectively (Grinfeld 1980, Abeyaratne 1983).

Consider a material curve $\mathbf{X} = \mathbf{X}(s)$ that lies on the interface in the reference configuration, where s parametrizes the curve. This curve is given by $\mathbf{x} = \mathbf{x}(\mathbf{X}(s))$ in the deformed configuration and a tangent vector to this curve is given by $\mathbf{F}\mathbf{X}'(s)$, where a prime denotes differentiation with respect to s. Since this tangent vector must be single-valued at the interface, we deduce that $[\mathbf{F}]\mathbf{A} = \mathbf{0}$ for all vectors $\mathbf{A}$ tangent to the interface in the reference configuration. It then follows that $[\mathbf{F}]$ must necessarily take the form

$$[\mathbf{F}] = \mathbf{f} \otimes \mathbf{N}, \tag{2.9}$$

where $\mathbf{f}$, defined by

$$\mathbf{f} = [\mathbf{F}]\mathbf{N}, \tag{2.10}$$

may be referred to as the amplitude of the jump in $[\mathbf{F}]$. With the use of (2.6) and (2.9), the jump condition (2.8) can be written as $([W] - \mathbf{f} \cdot \boldsymbol{\pi}\mathbf{N})\,\mathbf{N} = \mathbf{0}$. Thus, (2.8) yields only a single jump condition, namely,

$$[W] - \mathbf{f} \cdot \boldsymbol{\pi}\mathbf{N} = 0. \tag{2.11}$$

Any static two-phase deformation must satisfy the equilibrium equation (2.5) away from the interface and must satisfy the jump conditions (2.6), (2.9) and (2.11) across the interface.

Suppose that $\mathbf{F}^-$ is known and we wish to determine $\mathbf{F}^+$. The five unknowns f_1, f_2, f_3, N_1, N_2 are restricted by only four scalar equations obtained from (2.6) and (2.11) (note that $\mathbf{N}$ is a unit vector). Thus, for each fixed $\mathbf{F}^-$, we expect to obtain a family of solutions. We note, however, that since $\mathbf{N}$ will appear in our analysis through N_1^2, N_2^2, N_3^2, which must be non-negative, we anticipate that $\mathbf{F}^-$ cannot arbitrarily be specified and must belong to a so-called *phase transition zone* (PTZ) (Freidin and Chiskis 1994a, b). Only those deformation gradients that belong to the PTZ can appear as a deformation gradient on either side of a phase interface. We note that throughout our analysis the "+" and "−" phases take equal footing, and therefore any restrictions/properties derived for the "−" phase are equally valid for the "+" phase.

Define a function $w(\eta)$ through $w(\eta) = W(\mathbf{F}^- + \eta[\mathbf{F}])$. It then follows from (2.4), (2.9) and (2.11) that $w'(0) = w'(1) = [W] = w(1) - w(0)$. Thus, by the mean-value theorem, there must exist an $\eta^* \in (0, 1)$ such that

$$w''(\eta^*) = 0, \tag{2.12}$$

which then implies that the strong ellipticity condition is necessarily violated at $\mathbf{F} = \mathbf{F}^- + \eta^*[\mathbf{F}]$ (Knowles and Sternberg 1978). However, a necessary condition for the stability of any deformation (and hence in particular the two-phase deformation) is that everywhere W is rank-one convex (Gurtin 1983). Throughout this paper we make the slightly stronger assumption that W is strongly elliptic (i.e. strict rank-one convex) on both sides of the phase boundary. Then $w''(0) > 0$, $w''(1) > 0$, and strong ellipticity must necessarily fail on some segment $\eta \in [\eta_1^*, \eta_2^*]$, where $0 < \eta_1^*, \eta_2^* < 1$. This will be reflected in the fact that a zone of non-ellipticity is always embedded in the PTZ.

The left Cauchy-Green strain tensor $\mathbf{B}$ is defined by $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and its three principal invariants are given by

$$\begin{aligned} I_1 &= \mathrm{tr}\,(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= J^2\,\mathrm{tr}\,\mathbf{B}^{-1} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \\ J &= \lambda_1\lambda_2\lambda_3, \end{aligned} \tag{2.13}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the three principal stretches. It is found to be more convenient to express the jump conditions (2.6), (2.9) and (2.11) in terms of the Cauchy stress tensor $\boldsymbol{\sigma} = J^{-1}\mathbf{F}\boldsymbol{\pi}^T$ and the unit normal $\boldsymbol{n}$ to the interface in the deformed configuration. We have

$$[\mathbf{F}] = (\mathbf{a}^- \otimes \mathbf{n})\mathbf{F}^- = (\mathbf{a}^+ \otimes \mathbf{n})\mathbf{F}^+, \tag{2.14}$$

$$[\boldsymbol{\sigma}]\mathbf{n} = \mathbf{0}, \tag{2.15}$$

$$[W] - \mathbf{c} \cdot \boldsymbol{\sigma}\mathbf{n} = 0, \tag{2.16}$$

where

$$\mathbf{c} = J^-\mathbf{a}^- = J^+\mathbf{a}^+, \quad \mathbf{a}^{\pm} = \frac{1}{|\mathbf{F}_{\pm}^T\mathbf{n}|}\mathbf{f}. \tag{2.17}$$

The continuity of $\mathbf{c}$ indicated above arises from the continuity of a surface area element on the interface (noting Nanson's formula $\mathbf{F}^T\mathbf{n}da = J\mathbf{N}dA$, where dA and da are two corresponding area elements in the undeformed and deformed configurations, respectively).

It will be seen that the normal $\mathbf{n}$ appears in our analysis through the *orientation invariants*

$$N_1 = \mathbf{n} \cdot \mathbf{B}\mathbf{n}, \quad N_{-1} = \mathbf{n} \cdot \mathbf{B}^{-1}\mathbf{n} \tag{2.18}$$

introduced by Freidin and Chiskis (1994a). These invariants are discontinuous across the interface, but two other alternative quantities G_1 and G_{-1}, defined later by Freidin *et al.* (2002) through

$$G_1 = \frac{N_1}{J^2}, \quad G_{-1} = \frac{I_2}{J^2} - N_{-1}, \tag{2.19}$$

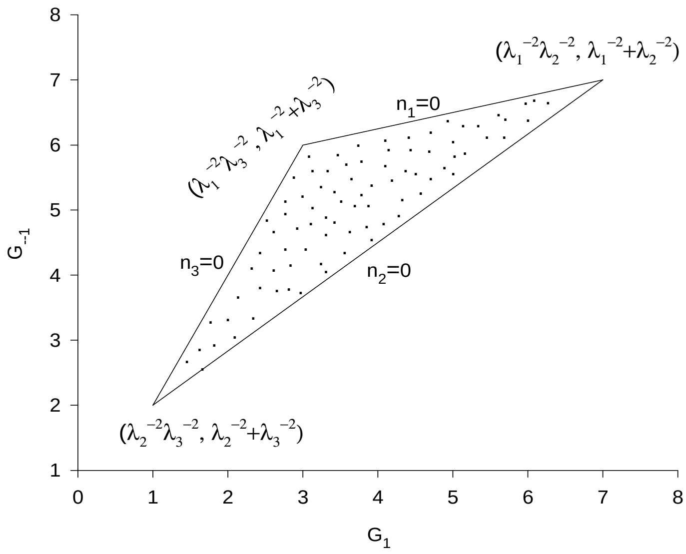

Figure 1. The region where solutions of (2.21) and (2.22) for n_1^2 and n_2^2 are non-negative.

are both continuous across the phase interface. That is

$$[G_1] = 0, \quad [G_{-1}] = 0. \tag{2.20}$$

The first condition is equivalent to the continuity condition indicated in (2.17), whilst the second follows from the continuity of line elements across the interface.

Relative to the principal axes of stretch, G_1 and G_{-1} take the form

$$G_1 = \lambda_1^{-2}\lambda_2^{-2} + n_1^2(\lambda_2^{-2}\lambda_3^{-2} - \lambda_1^{-2}\lambda_2^{-2}) + n_2^2(\lambda_1^{-2}\lambda_3^{-2} - \lambda_1^{-2}\lambda_2^{-2}), \tag{2.21}$$

$$G_{-1} = \lambda_1^{-2} + \lambda_2^{-2} + n_1^2(\lambda_3^{-2} - \lambda_1^{-2}) + n_2^2(\lambda_3^{-2} - \lambda_2^{-2}). \tag{2.22}$$

In Fig. 1 we have shown the region in the G_1G_{-1}-plane where the solutions of (2.21) and (2.22) for n_1^2, n_2^2 are non-negative. This diagram of admissible values plays an important role in the construction of PTZs in the general case (see the discussion around (3.42)).

For plane-strain problems, we have $\lambda_3 = 1, n_3 = 0$ and equations (2.21) and (2.22) reduce to

$$G_1 = G_{-1} - 1 = \lambda_1^{-2} + \left(\lambda_2^{-2} - \lambda_1^{-2}\right) n_1^2. \tag{2.23}$$

The n_1^2 will satisfy the inequalities $0 \le n_1^2 \le 1$ only if (G_1, G_{-1}) lies on the side $n_3 = 0$ of the triangle in Fig. 1.

3. A piecewise-homogeneous state

We now specialize to the case when the elastic body occupies the infinite space. We assume that under a plane-strain homogeneous deformation, a phase transition takes place and the infinite space divides into two half-spaces of different phases with a plane interface located at $\mathbf{N} \cdot \mathbf{X} = 0$. Thus,

$$\mathbf{F} = \begin{cases} \mathbf{F}^- & \text{in } 0 < \mathbf{N}\cdot\mathbf{X} < \infty \\ \mathbf{F}^+ & \text{in } -\infty < \mathbf{N}\cdot\mathbf{X} < 0. \end{cases} \tag{3.1}$$

Without loss of generality, we may assume that the principal axes of stretch of $\mathbf{F}^-$ coincide with the X_1- and X_2-axes. Thus, we may write

$$\mathbf{F}^- = \begin{pmatrix} \lambda_1^- & 0 \\ 0 & \lambda_2^- \end{pmatrix}, \tag{3.2}$$

where λ_1^- and λ_2^- are positive constants. We note that for the present plane-strain problem, we have only three unknowns f_1, f_2, N_1^2 that are determined by three equations deduced from (2.6) and (2.11). This is in contrast with the general 3-D problem where we have an under-determined system of five unknowns determined by four equations. The three equations for f_1, f_2, N_1^2 will have a solution satisfying $0 \le N_1^2 \le 1$ only if $(\lambda_1^-, \lambda_2^-)$ lies in a PTZ (see discussions following equation (3.49)). Only a subset of this PTZ (which will be referred to as a restricted PTZ) will correspond to a special choice of N_1 such as $N_1 = 0$.

To illustrate how a PTZ is constructed and the role it plays in the understanding of a phase transformation process, we first consider the simple case when $N_1 = 0, N_2 = 1$. From Nanson's formula we deduce that $n_i = \delta_{2i}$. It can then be shown with the aid of the continuity condition $(2.20)_1$ that $[\lambda_1] = 0$ and that $\mathbf{F}^+$ is also diagonal: $\mathbf{F}^+ = \mathrm{diag}\left\{\lambda_1^-, \lambda_2^+, \right\}$, where the principal stretch λ_2^+ is to be determined. Thus

$$[\mathbf{F}] = \mathrm{diag}\,\{0, [\lambda_2]\}\,, \quad \mathbf{f} = (0, [\lambda_2])^T. \tag{3.3}$$

For the present problem, (2.4) reduces to

$$\boldsymbol{\pi} = \mathrm{diag}\left\{\frac{\partial W}{\partial \lambda_1}, \frac{\partial W}{\partial \lambda_2}\right\}, \tag{3.4}$$

and the jump conditions (2.6) and (2.11) reduce to

$$[\frac{\partial W}{\partial \lambda_2}] = 0, \tag{3.5}$$

and

$$[W] - [\lambda_2]\frac{\partial W}{\partial \lambda_2} = 0. \tag{3.6}$$

We may solve (3.5) to express λ_2^+ in terms of λ_1^- and λ_2^-. Equation (3.6) then yields a single equation for λ_1^- and λ_2^-, which defines a curve in the $\lambda_1\lambda_2$-plane. This curve can be viewed as a restricted PTZ. Clearly, for any chosen λ_1^-, a phase transformation is possible only if this equation has at least two positive roots for λ_2^-. If there are exactly two roots, then one root is λ_2^-, whilst the other root is λ_2^+.

To obtain more explicit results, we now consider the Hadamard strain-energy function given by

$$W = \frac{1}{2}cI_1 + \frac{1}{2}dI_2 + \phi(J), \tag{3.7}$$

where c and d are constants. We further assume that ϕ takes the simple form

$$\phi(J) = F_c + \frac{1}{4}(J - J_c)^4 - \frac{1}{2}a_1(J - J_c)^2 + a_0(J - J_c), \tag{3.8}$$

where F_c, J_c, a_0, a_1 are also constants. For the 3D-case (3.7) and (3.8) were examined by Freidin and Chiskis (1994a). In all the subsequent illustrative calculations, we take $a_1 = 0.8, d = 0.2, J_c = 2, c = 0.032$.

For the plane-strain deformation under consideration, equation (2.12) dictates that $\partial^2 W/\partial \lambda_2^2$, given by

$$\frac{\partial^2 W}{\partial \lambda_2^2} = \frac{\partial \pi_{22}}{\partial \lambda_2} = c + d + \lambda_1^2\left\{d - a_1 + 3(J - J_c)^2\right\}, \tag{3.9}$$

must vanish for some (λ_1, λ_2). Thus, the material and λ_1 must be such that

$$c + d + \lambda_1^2(d - a_1) \le 0. \tag{3.10}$$

Such considerations are useful when choosing strain-energy functions and material constants that will actually give rise to phase transformations.

The jump conditions (3.5) and (3.6) now reduce to

$$\{c + d(\lambda_1^2 + 1)\}[\lambda_2] + [\phi'(J)]\lambda_1 = 0, \tag{3.11}$$

$$\{c + d(\lambda_1^2 + 1)\}\left\{\frac{1}{2}[\lambda_2^2] - \lambda_2^-[\lambda_2]\right\} + [\phi] - \lambda_1[\lambda_2]\phi'(J^-) = 0, \tag{3.12}$$

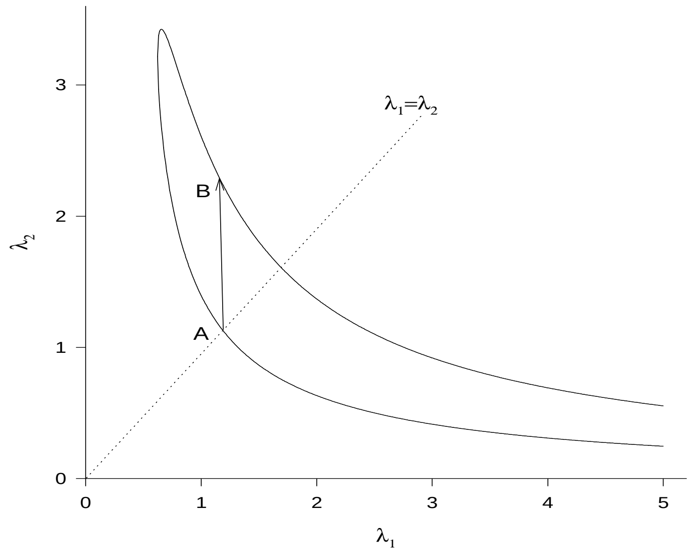

Figure 2. The PTZ curve given by a typical solution of (3.17).

where we have omitted the superscript "−" on λ_1 since it is continuous across the interface. On eliminating $c + d(\lambda_1^2 + 1)$ from (3.11) and (3.12), we obtain

$$\frac{[\phi]}{[J]} = \frac{1}{2}(\phi'_+ + \phi'_-). \tag{3.13}$$

On substituting (3.8) into (3.13), we obtain

$$J_+ + J_- = 2J_c, \tag{3.14}$$

or equivalently,

$$\lambda_2^+ = \frac{1}{\lambda_1}(2J_c - \lambda_1\lambda_2^-). \tag{3.15}$$

With the use of (3.8) and (3.14), equation (3.11) becomes

$$d + \frac{c+d}{\lambda_1^2} + (J_- - J_c)^2 - a_1 = 0, \tag{3.16}$$

which may be solved for λ_2^- to give

$$\lambda_2 = \frac{1}{\lambda_1}\left\{ J_c \pm \sqrt{a_1 - d - \frac{c+d}{\lambda_1^2}} \right\}, \tag{3.17}$$

where we have omitted the superscript "−" on λ_2 since this result is valid for both phases. It is seen that the strict form of the necessary condition (3.10) is also sufficient for a phase transformation to be possible. When the material constants a_1, c, d are fixed, equation (3.17) shows that a phase transformation becomes possible when

$$\lambda_1^2 > \frac{c+d}{a_1 - d}. \tag{3.18}$$

For each λ_1 satisfying the above inequality, there are exactly two values of λ_2 which gives the values of λ_2 on the two sides of the interface. We also observe that if, for instance, the infinite elastic body is subjected to a biaxial stretch, then as $\lambda_1 = \lambda_2$ is increased gradually, a phase transformation will take place when the line $\lambda_1 = \lambda_2$ intersects the lower branch of the curve given by (3.17) (point A in Fig. 2). A new phase forms and across the interface between the two phases the principal stretch λ_2 jumps from the value at point A to the value at point B.

The curve shown in Fig.2 is a typical PTZ restricted by the assumptions $N_A = \delta_{2A}$, $\lambda_3 \equiv 1$. Only those values of λ_1 and λ_2 on the curve can appear as values of principal stretches on either side of the interface. In the above example the orientation of the interface is specified beforehand. It will be shown next that when the orientation of the interface is not specified, the PTZ is not a curve but an area.

We now relax the assumption $N_A = \delta_{2A}$ and the assumption of plane-strain. We shall relate the above analysis to a more general framework that has recently been developed by Freidin and Chiskis (1994a, b). See also Freidin *et al.* (2002).

As stated in Section 2, in the most general case the five unknowns $f_1, f_2, f_3,\ N_1^2, N_2^2$ are determined by four scalar jump conditions obtained from (2.6) and (2.11), or equivalently from (2.15) and (2.16). For an isotropic elastic material, the strain-energy function may be assumed to take the form

$$W = W(I_1, I_2, J), \tag{3.19}$$

where I_1, I_2, and J are given by (2.13). In this case it turns out that it is more convenient to assume that I_1^-, I_2^-, J^- are specified, and to take $I_1^+, I_2^+, J^+, G_1, G_{-1}$ instead to be the five unknowns. Following Freidin and Chiskis (1994a, b), our immediate objective is to express the four jump conditions (2.15) and (2.16) in terms of these five unknowns. Once this has been achieved, we would solve three of these jump conditions for I_1^+, I_2^+, J^+ and substitute the resulting expressions into the fourth jump conditions to obtain (3.42).

To evaluate (2.15), we need an expression of $\boldsymbol{\sigma}$. With the use of (2.4) and (3.19), we find

$$\boldsymbol{\sigma} = \mu_0 \boldsymbol{I} + \mu_1 \boldsymbol{B} + \mu_{-1}\mathbf{B}^{-1}, \tag{3.20}$$

where

$$\mu_0 = W_3 + 2J^{-1}I_2W_2, \quad \mu_1 = 2J^{-1}W_1, \quad \mu_{-1} = -2JW_2, \tag{3.21}$$

and $W_1 = \partial W/\partial I_1, W_2 = \partial W/\partial I_2, W_3 = \partial W/\partial J$.

We define a projector tensor $\mathbf{P}$ through

$$\mathbf{P} = \mathbf{I} - \mathbf{n}\otimes\mathbf{n}. \tag{3.22}$$

The jump condition (2.15) is then equivalent to

$$\mathbf{n}\cdot[\boldsymbol{\sigma}]\mathbf{n} = 0, \quad \text{and} \quad \mathbf{P}[\boldsymbol{\sigma}]\mathbf{n} = \mathbf{0}. \tag{3.23}$$

With the use of (3.20) and (3.21), equation $(3.23)_{1,2}$ may be reduced to

$$[W_3] + 2G_{-1}[JW_2] + 2G_1[JW_1] = 0, \tag{3.24}$$

$$[W_1\mathbf{t}_1] - [W_2\mathbf{t}_2] = 0, \tag{3.25}$$

where

$$\mathbf{t}_1 = \frac{1}{J}\mathbf{PBn}, \quad \mathbf{t}_2 = J\mathbf{PB}^{-1}\mathbf{n}. \tag{3.26}$$

Equation (3.24) is already of the required form. To simplify (3.25) further, we would need expressions of $\mathbf{B}^{\pm}$ and $\mathbf{B}_{\pm}^{-1}$. With the aid of (2.14) and the results

$$\det(\mathbf{I}+\mathbf{a}\otimes\mathbf{n}) = 1+\mathbf{a}\cdot\mathbf{n}, \quad (\mathbf{I}+\mathbf{a}\otimes\mathbf{n})^{-1} = \mathbf{I} - (1+\mathbf{a}\cdot\mathbf{n})^{-1}\mathbf{a}\otimes\mathbf{n},$$

we find

$$\frac{J^+}{J^-} = 1 + \mathbf{a}^-\cdot\mathbf{n}, \tag{3.27}$$

$$\mathbf{B}^+ = \mathbf{B}^- + \mathbf{a}^-\otimes\mathbf{B}^-\mathbf{n} + \mathbf{B}^-\mathbf{n}\otimes\mathbf{a}^- + N_1^-\mathbf{a}^-\otimes\mathbf{a}^-, \tag{3.28}$$

$$\mathbf{B}_+^{-1} = \mathbf{B}_-^{-1} - \frac{J^-}{J^+}\left(\mathbf{n}\otimes\mathbf{B}_-^{-1}\mathbf{a}^- + \mathbf{B}_-^{-1}\mathbf{a}^-\otimes\mathbf{n}\right) + \frac{J_-^2}{J_+^2}(\mathbf{a}^-\cdot\mathbf{B}_-^{-1}\mathbf{a}^-)\mathbf{n}\otimes\mathbf{n}. \tag{3.29}$$

It then follows that

$$\mathbf{t}_1^+ = \mathbf{t}_1^- + G_1\mathbf{h}, \quad \mathbf{t}_2^+ = \mathbf{t}_2^- - \mathbf{PB}^{-1}\mathbf{h}, \tag{3.30}$$

where $\mathbf{h} = \mathbf{Pc}$ and use has been made of the relation $\mathbf{a}^- = \mathbf{c}/J^-$ and the decomposition

$$\mathbf{c} = (\mathbf{c}\cdot\mathbf{n})\mathbf{n} + \mathbf{h} = J^-(\mathbf{a}^-\cdot\mathbf{n})\mathbf{n} + \mathbf{h} = [J]\mathbf{n} + \mathbf{h}. \tag{3.31}$$

On substituting (3.30) into (3.25) and making use of the identity $[fg] = f^+[g] + [f]g^-$, we obtain

$$(W_1^+ G_1 \mathbf{I} + W_2^+ \mathbf{P}\mathbf{B}^{-1})\mathbf{h} = [W_2]\mathbf{t}_2^- - [W_1]\mathbf{t}_1^-. \tag{3.32}$$

This equation is used to express $\mathbf{h}$ in terms of the five unknowns $I_1^+, I_2^+, J^+, G_1, G_{-1}$ (and other "$-$" quantities which are assumed to be known).

It can be shown that the determinant of the coefficient matrix of $\mathbf{h}$ in (3.32) is equal to $G_1^2 W_1^+ D^+$, where

$$D^+ = G_1 (W_1^+)^2 + G_{-1} W_1^+ W_2^+ + (W_2^+)^2. \tag{3.33}$$

Equation (3.32) then shows that provided $D_+ \neq 0$, the component $\mathbf{h}$ of $\mathbf{c}$ can be expressed as a linear combination of $\mathbf{t}_2^-$ and $\mathbf{t}_1^-$. We note that at the three vertices of the triangular region in Fig. 1, D^+ takes the form

$$D^+ = \lambda_{i+}^{-2} \lambda_{j+}^{-2} \left(W_1^+ + \lambda_{i+}^2 W_2^+ \right) \left(W_1^+ + \lambda_{j+}^2 W_2^+ \right),$$

where $(i, j) = (2, 3), (1, 3)$ or $(1, 2)$. This expression is positive when the Baker-Ericksen inequalities $W_1 + \lambda_k^2 W_2 > 0 (k = 1, 2, 3)$ are satisfied. Since D^+ is linear in G_1 and G_{-1}, it follows that D^+ is positive over the entire triangular region in Fig. 1 (and hence for all the admissible values of G_1 and G_{-1}) provided the strong ellipticity condition, and hence the Baker-Ericksen inequalities, are satisfied.

Thus, equation $(3.23)_2$ does not yield two equations for I_1^+, I_2^+, J^+, G_1 and G_{-1} directly. Instead, it gives us an expression for $\mathbf{h}$. The two desired equations are obtained from (3.27)–(3.29) as

$$[I_1] = G_1([J^2] + \mathbf{h} \cdot \mathbf{h}) + 2\mathbf{h} \cdot \mathbf{t}_1^-, \tag{3.34}$$

$$[I_2] = G_{-1}[J^2] - 2\mathbf{h} \cdot \mathbf{t}_2^- + \mathbf{h} \cdot \mathbf{B}_-^{-1}\mathbf{h}, \tag{3.35}$$

where $\mathbf{h}$ is now known.

Finally, with the use of (3.20), (3.28) and (3.29), the jump condition (2.16) gives us the fourth equation

$$[W] = \tau_n [J] + 2W_1^- \mathbf{h} \cdot \mathbf{t}_1^- - 2W_2^- \mathbf{h} \cdot \mathbf{t}_2^-, \tag{3.36}$$

where

$$\tau_n = \mathbf{n} \cdot \boldsymbol{\sigma} \mathbf{n} = 2J(G_1 W_1 + G_{-1} W_2) + W_3. \tag{3.37}$$

The four equations, namely (3.24), (3.34), (3.35) and (3.36), can be solved for the five unknowns $G_1, G_{-1}, I_1^+, I_2^+, J^+$. As an example, consider the special case when $W = W(I_1, J)$. Equation (3.32) gives $W_1^+ G_1 \mathbf{h} = -[W_1]\mathbf{t}_1^-$, and these four equations are given by

$$[W_3] + 2G_1[JW_1] = 0, \tag{3.38}$$

$$[W] = \frac{W_1^+ W_3^- + W_1^- W_3^+}{W_1^+ + W_1^-}[J] + \frac{2W_1^+ W_1^-}{W_1^+ + W_1^-}[I_1], \tag{3.39}$$

$$[I_1] = G_1[J^2] - \frac{[W_1^2]}{W_{1+}^2}(I_1^- - G_1^{-1} G_{-1} - G_1 J_-^2), \tag{3.40}$$

$$[I_2] = G_{-1}[J^2] - \frac{[W_1^2]}{W_{1+}^2}(I_2^- - J_-^2 G_{-1} - G_1^{-1}). \tag{3.41}$$

If we solve three of these four equations to obtain

$$\begin{aligned} I_1^+ &= I_1^+ \left(G_1, G_{-1} | I_1^-, I_2^-, J_- \right), \\ I_2^+ &= I_2^+ \left(G_1, G_{-1} | I_1^-, I_2^-, J_- \right), \\ J^+ &= J^+ \left(G_1, G_{-1} | I_1^-, I_2^-, J_- \right), \end{aligned}$$

then the fourth equation defines a curve in the $G_1 G_{-1}$-plane:

$$\psi(G_1, G_{-1} | I_1^-, I_2^-, J^-) = 0. \tag{3.42}$$

For each fixed set of (I_1^-, I_2^-, J^-), phase transition is possible only if the curve given by (3.42) intersects the triangular region shown in Fig. 1. Thus, in the most general case the PTZ is defined by

$$\text{PTZ} := \big\{(I_1^-, I_2^-, J^-) \in \mathbf{R}^3 |\ I_1^- > 0, I_2^- > 0, J^- > 0, \text{and the intersection between}$$
$$\text{(3.42) and the triangular region in Fig. 1 is non-empty}\ \big\}.$$

For the special class of material given by (3.7), $[W_1] = [W_2] = 0$, and equation (3.32) then yields $\mathbf{h} = \mathbf{0}$. Equations (3.36), (3.24), (3.34) and (3.35) reduce to

$$\frac{[\phi]}{[J]} = \frac{1}{2}(\phi'_+ + \phi'_-), \tag{3.43}$$

$$\frac{[\phi']}{[J]} = -(cG_1 + dG_{-1}), \tag{3.44}$$

$$[I_1] = G_1[J^2], \quad [I_2] = G_{-1}[J^2], \tag{3.45}$$

where in obtaining the simplified form in (3.43), we have also made use of the expressions (3.44) and (3.45).

Equation (3.43) can be solved to express J^+ in terms of J^-. With J^+ eliminated, the left hand side of (3.44) is a function of J^- alone so that (3.44) takes the form (3.42). We note that if ϕ is a quartic polynomial, then (3.43) can be reduced to

$$J^+ = J^- - 2\phi'''(J^-)/\phi^{(4)}(J^-), \tag{3.46}$$

which recovers (3.14) when ϕ is given by (3.8).

When ϕ is given by (3.8), equations (3.44), and hence (3.42), take the form

$$cG_1 + dG_{-1} + (J^- - J_c)^2 - a_1 = 0. \tag{3.47}$$

In the space of principal stretches, the PTZ is formed by all sets of principal stretches at which (3.47) has a non-empty intersection with the shaded triangular region in Fig. 1. It then follows that the PTZ is bounded by the three surfaces obtained by requiring (3.47) to intersect the three vertices in Fig. 1 respectively. Thus, the three bounding surfaces are

$$\begin{aligned} &\frac{c}{\lambda_2^2\lambda_3^2} + d(\frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}) + (\lambda_1\lambda_2\lambda_3 - J_c)^2 - a_1 = 0, \\ &\frac{c}{\lambda_1^2\lambda_3^2} + d(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_3^2}) + (\lambda_1\lambda_2\lambda_3 - J_c)^2 - a_1 = 0, \\ &\frac{c}{\lambda_1^2\lambda_2^2} + d(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) + (\lambda_1\lambda_2\lambda_3 - J_c)^2 - a_1 = 0. \end{aligned} \tag{3.48}$$

We observe that on these three bounding surfaces, we have $n_1^2 = 1, n_2^2 = 1, n_3^2 = 1$, respectively. For plane-strain deformations, the PTZ boundaries can be obtained by setting $\lambda_3 = 1$ in (3.48). The third surface in (3.48) should be neglected since it corresponds to $n_3^2 = 1$. Alternatively, the PTZ can be obtained from those of (λ_1, λ_2) at which (3.47) with $\lambda_3 = 1$ has a non-empty intersection with the side $n_3 = 0$ of the triangle in Fig. 1. Thus, again the PTZ is bounded by $(3.48)_{1,2}$ with $\lambda_3 = 1$.

The PTZ for plane-strain deformations can also be constructed directly as follows. On substituting (2.23) into (3.47), we obtain

$$(J - J_c)^2 - a_1 + d + (c+d)\left\{\lambda_1^{-2} + (\lambda_2^{-2} - \lambda_1^{-2})n_1^2\right\} = 0, \tag{3.49}$$

where we have omitted the superscripts "$-$" on λ_1, λ_2 and J since it should also apply to the "+" phase. The PTZ boundaries clearly correspond to $n_1^2 = 0, 1$. When $n_1^2 = 0$, (3.49) yields

$$\lambda_2 = \frac{1}{\lambda_1}\left\{J_c \pm \sqrt{a_1 - d - \frac{c+d}{\lambda_1^2}}\right\}, \tag{3.50}$$

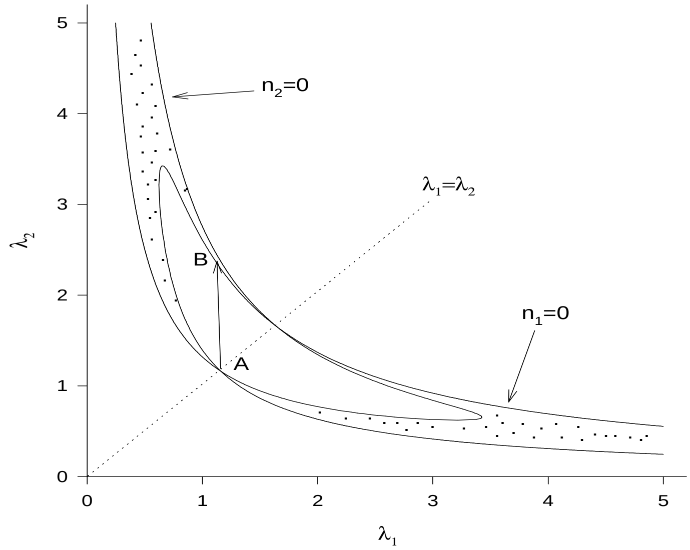

Figure 3. The PTZ (shaded region) with boundaries defined by (3.50) and (3.51) when $a_1 = 0.8, d = 0.2, J_c = 2, c = 0.032$.

which is the same as (3.17) and $(3.48)_2$. When $n_1^2 = 1$, (3.49) yields

$$\lambda_1 = \frac{1}{\lambda_2}\left\{J_c \pm \sqrt{a_1 - d - \frac{c+d}{\lambda_2^2}}\right\}, \tag{3.51}$$

which is the same as $(3.48)_1$. The curve corresponding to (3.51) can be obtained by reflecting the curve (3.50) about the line $\lambda_1 = \lambda_2$. It is easily found that $0 \le n_1^2 \le 1$ corresponds to the shaded region in Fig. 3, which is the PTZ for the case under consideration.

The pair of principal stretches at any point inside the shaded region in Fig. 3 qualifies as a pair of principal stretches on one side of a phase boundary. These principal stretches must be a solution of (3.49) for some n_1^2 satisfying $0 < n_1^2 < 1$. The principal stretches appearing on the other side of the phase boundary are then determined by solving (3.49) together with $J^+ = 2J_c - J^-$. As an example, we take $\lambda_1^- = 4, \lambda_2^- = 0.45$. Equation (3.49) then yields $n_1^2 = 0.48224$. Now solving (3.49) with $\lambda_1\lambda_2 = 2J_c - 4 \times 0.45$, we find that there are two solutions for the principal stretches on the other side of the phase boundary: $\lambda_1^+ = 4.89996, \lambda_2^+ = 0.448983$, or $\lambda_1^+ = 0.465225, \lambda_2^+ = 4.7289$.

It can easily be shown that when $0 < n_1^2 < 1$, the curve given by (3.49) is necessarily closed. Fig. 4 shows this curve when $n_1 = 0.5$. It also shows that for every point on this curve, A say, there are two other points B and C on this curve that satisfy the additional condition $J^+ = 2J_c - J^-$.

Fig. 3 shows that any loading path starting from the stress-free state will hit the $n_1 = 0$ curve or $n_2 = 0$ curve first. This means, for example, that under biaxial stretching the new phase boundary is either perpendicular or parallel to one of the principal axes. We remark, however, that it is possible to find strain-energy functions for which the boundaries of the PTZ do not correspond to either $n_1 = 0$ or $n_2 = 0$. In this case the new phase boundary under biaxial stretching would be skewed with respect to the principal axes (as in the case of shear band formations). Such examples can be found in Freidin and Chiskis (1994b), Freidin *et al.* (2002).

4. Incremental equations

The *primary* two-phase deformation determined in the previous section is a stationary point of the energy functional with respect to perturbations of (i) the displacement field, (ii) the phase boundary in the current configuration, and (iii) the phase boundary in the reference configuration (as ensured, respectively, by the satisfaction of the equilibrium equation (2.5) and the jump conditions (2.6) and (2.8)). However, it is not clear whether such a two-phase deformation is stable, despite the fact that rank-one convexity ensures stability with respect to a special class of perturbations (known as Weierstrass-type perturbations, see Cherkaev 1991, p.151). We shall use two stability criteria to investigate its stability properties. The first one is a dynamic approach where we apply a perturbation and determine if the perturbation will grow. The other one is to determine whether

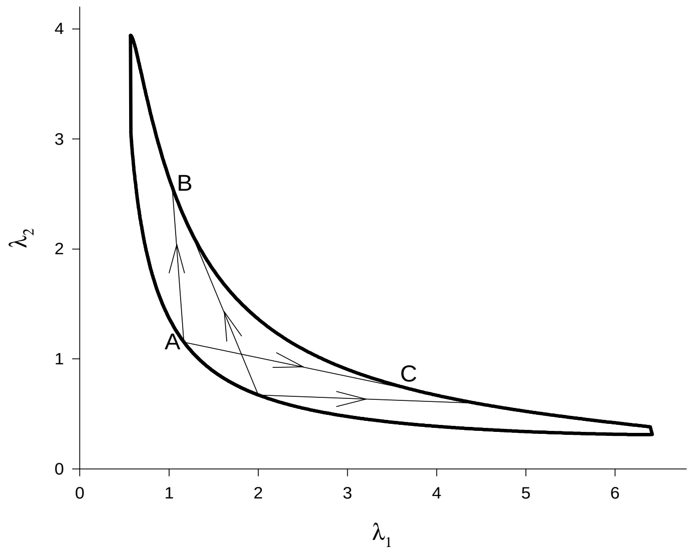

Figure 4. The curve (thick line) given by (3.49) when $n_1 = 0.5$. If principal stretches on the $-$ side of an interface are given by the coordinates of A, then the principal stretches on the $+$ side of the interface are given by the coordinates of either B or C

the two-phase state is an energy minimizer. In both approaches, the perturbation takes the form of a normal mode. In this section, we shall derive the incremental equations governing the evolution of such a normal mode.

Originally, the phase boundary is given by $\mathbf{N}\cdot\mathbf{X} = 0$. For convenience, we shall use a parametric representation

$$\mathbf{X} = \mathbf{Y}(s), \quad -\infty < s < \infty, \quad \text{with} \quad Y_1(s) = N_2 s, \quad Y_2(s) = -N_1 s. \tag{4.1}$$

Suppose that under the same dead-load boundary conditions at infinity, the interface of the two-phase state is subjected to a small amplitude perturbation and its new location is given by

$$\mathbf{X} = \mathbf{Y}(s) + \boldsymbol{\xi}(s,t), \quad \text{where } |\boldsymbol{\xi}| \ll 1. \tag{4.2}$$

With the interface perturbed, although the primary deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$ still satisfies the equilibrium equation and boundary conditions at infinity, it will in general not satisfy the jump conditions (2.6) and (2.11) now applied at the new interface (4.2). Let $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{X},t)$ be the new solution that satisfies the jump condition (2.6) as well as the equilibrium equation, but not necessarily the jump condition (2.11). Thus, we shall neglect the inertial term in the equation of motion, and following Eremeyev *et al.* (2003), we shall replace (2.11) by

$$\frac{\partial \Gamma}{\partial t} = -k([W] - \mathbf{f}\cdot\boldsymbol{\pi}\tilde{\mathbf{N}}), \quad \Gamma \equiv \boldsymbol{\xi}\cdot\mathbf{N} \tag{4.3}$$

where k is a positive constant, $\partial\Gamma/\partial t$ represents the normal speed of the interface, and $\tilde{\mathbf{N}}$ is the unit normal to the perturbed interface (4.2). This approach may be referred to as a *quasi-static* approach. The evolution equation is motivated by the fact that the rate of energy dissipated as the interface traverses the material is given by (Knowles and Sternberg 1978, Knowles 1979)

$$\int ([W] - \mathbf{f}\cdot\boldsymbol{\pi}\tilde{\mathbf{N}})\frac{\partial \Gamma}{\partial t} ds.$$

The assumption (4.3) is made to ensure that this integral is non-negative.

In a normal mode approach, we consider perturbations of the form

$$\Gamma(s,t) = \gamma(t)e^{i\mathbf{K}\cdot\mathbf{X}} = \gamma(t)e^{is}, \tag{4.4}$$

where $\gamma(t)$ is the amplitude function to be determined and the $\mathbf{K}$ is a unit vector orthogonal to $\mathbf{N}$:

$$\mathbf{K} = (N_2, -N_1)^T. \tag{4.5}$$

In the representation (4.4) the wavenumber along the interface is chosen to be unity. This is without loss of generality since the present problem does not have a natural lengthscale. The above choice of unit wavenumber amounts to using the inverse of the actual wavenumber as the reference lengthscale.

The quasi-static deformation $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{X}, t)$ is essentially the solution to a joint-body problem (i.e. two half-spaces of different materials joined together along the interface (4.2)) and contains $\gamma(t)$ as a parametric function. On linearizing the right hand side of (4.3), we would obtain a linear differential equation of the form

$$-\frac{1}{k}\frac{d\gamma}{dt} = L\gamma, \tag{4.6}$$

where the coefficient L contains all the information about the primary two-phase deformation determined in the previous section. It will be shown later that L is always real. Our kinetic stability criterion may then be stated as follows. The primary two-phase state is unstable if $L < 0$ in which case γ grows exponentially and stable if $L > 0$. The case $L = 0$ corresponds to marginal/neutral stability.

We write

$$\tilde{\mathbf{x}} = \mathbf{x}(\mathbf{X}) + \mathbf{u}(\mathbf{X}, t), \tag{4.7}$$

where in the context of the above discussions $\mathbf{u}(\mathbf{X}, t)$ may be viewed as the incremental displacement field induced by the interface perturbation $\boldsymbol{\xi}$. We remark, however, that $\mathbf{u}$ may also be superimposed on the primary deformation independently of $\boldsymbol{\xi}$. We now derive the incremental equations satisfied by $\mathbf{u}$. To simplify notation, from now on we shall suppress the dependence of $\mathbf{u}$ on t.

First, linearizing the expression for the location of the interface in the current configuration, we obtain

$$\begin{aligned} \tilde{\mathbf{x}} &= \mathbf{x}(\mathbf{Y}(s) + \boldsymbol{\xi}) + \mathbf{u}(\mathbf{Y}(s) + \boldsymbol{\xi}), \\ &= \mathbf{x}(\mathbf{Y}(s)) + \bar{\mathbf{F}}(\mathbf{Y}(s))\boldsymbol{\xi} + \mathbf{u}(\mathbf{Y}(\mathbf{s})), \end{aligned} \tag{4.8}$$

where here and hereafter we neglect terms that are quadratic or of higher order in Γ and an overbar signifies association with the primary two-phase deformation determined in the previous section. Since this expression should be independent of which side of the interface it is evaluated, we have

$$[\bar{\mathbf{F}}]\boldsymbol{\xi} + [\mathbf{u}] = \mathbf{0}, \tag{4.9}$$

where the jumps are evaluated at the unperturbed interface. The deformation gradient corresponding to (4.7) is

$$\mathbf{F} = \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \bar{\mathbf{F}} + \mathbf{V}, \quad \text{where} \quad \mathbf{V} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}. \tag{4.10}$$

Linearizing the stress tensor given by (2.4) about $\mathbf{F} = \bar{\mathbf{F}}$, we obtain

$$\pi_{iA} = \bar{\pi}_{iA} + E_{iAjB} u_{j,B}, \tag{4.11}$$

where

$$E_{iAjB} = \left. \frac{\partial^2 W}{\partial F_{iA} \partial F_{jB}} \right|_{\mathbf{F}=\bar{\mathbf{F}}}. \tag{4.12}$$

Thus, the incremental equilibrium equation is simply

$$E_{iAjB} u_{j,BA} = 0. \tag{4.13}$$

Evaluating the expression of $\mathbf{F}$ given by (4.10) on the immediate "$-$" and "$+$" sides of the interface (4.2), linearizing and then subtracting the two expressions, we obtain

$$[\mathbf{F}] = [\bar{\mathbf{F}}] + [\mathbf{V}]. \tag{4.14}$$

We emphasize that the jump on the left hand of (4.14) is evaluated at the perturbed interface but the two jumps on the right hand sides are evaluated at the unperturbed interface and (4.14) is only valid to leading order. In obtaining (4.14) use has also been made of the fact that $\bar{\mathbf{F}}$ is homogeneous in both phases. Applying the same procedure to (4.11), we obtain

$$\pi^-_{iA} = \bar{\pi}^-_{iA} + E^-_{iAjB} u^-_{j,B}, \tag{4.15}$$

$$[\pi_{iA}] = [\bar{\pi}_{iA}] + [E_{iAjB}u_{j,B}]. \tag{4.16}$$

We now evaluate the jump condition (2.6). First, the unit normal $\tilde{\mathbf{N}}$ is given by

$$\tilde{\mathbf{N}} = \mathbf{N} - (\mathbf{N}\cdot\boldsymbol{\xi}')\mathbf{K} = \mathbf{N} - \Gamma'\mathbf{K} \tag{4.17}$$

to leading order, where a prime denotes differentiation with respect to the parameter s. Thus, the jump condition (2.6) yields

$$[E_{iAjB}u_{j,B}]N_A - [\bar{\pi}_{iA}]K_A\Gamma' = 0. \tag{4.18}$$

The incremental displacement field $\mathbf{u}$ is determined by solving the equilibrium equation (4.13) subjected to the continuity conditions (4.9), (4.18) and the decay conditions $\mathbf{u} \to 0$ as $X_2 \to \pm\infty$. We observe that this is essentially a joint-body problem with Γ appearing in the forcing terms. As will become clear shortly, a solution can be found for each Γ unless the two bonded half-spaces are only neutrally stable with respect to interfacial perturbations (in that case the interface can support a standing wave and so $\mathbf{u}$ cannot be uniquely determined).

We now proceed to linearize equation (4.3). First, we have

$$W = W(\bar{\mathbf{F}}) + \bar{\pi}_{iA}u_{i,A}. \tag{4.19}$$

It then follows that

$$[W] = [\bar{W}] + [\bar{\pi}_{iA}u_{i,A}], \tag{4.20}$$

where $\bar{W} = W(\bar{\mathbf{F}})$. The amplitude vector $\mathbf{f}$ is computed according to (2.10). Thus, with the use of (4.14) and (4.17), we have

$$f_i = \bar{f}_i + [u_{i,A}]N_A, \tag{4.21}$$

where

$$\bar{f}_i = [\bar{F}_{iA}]N_A \tag{4.22}$$

is the amplitude of $[\bar{F}_{iA}]$ associated with the primary deformation. The factor $\boldsymbol{\pi}\tilde{\mathbf{N}}$ in (4.3) can be computed with the aid of (4.15) and (4.17) and is given by

$$\pi_{iA}N_A = \pi^-_{iA}N_A = \bar{\pi}_{iA}N_A + E^-_{iAjB}u^-_{j,B}N_A - \bar{\pi}^-_{iA}K_A\Gamma'. \tag{4.23}$$

Finally, on substituting (4.20), (4.21) and (4.23) into (4.3), we obtain

$$-\frac{1}{k}\frac{\partial\Gamma}{\partial t} = [\bar{\pi}_{iA}u_{i,A}] - \bar{f}_i\left(E^-_{iAjB}u^-_{j,B}N_A - \bar{\pi}^-_{iA}K_A\Gamma'\right) - \bar{\pi}_{iB}[u_{i,A}]N_BN_A, \tag{4.24}$$

where use has been made use of the fact that

$$[\bar{W}] - \bar{\mathbf{f}}\cdot\bar{\boldsymbol{\pi}}\mathbf{N} = 0, \tag{4.25}$$

which is simply (2.11) applied to the primary deformation.

We observe that if we set $\partial\Gamma/\partial t = 0$, then the equations derived in this section are the governing equations for an adjacent equilibrium state. If these static equations have a non-trivial solution, then the primary two-phase state may cease to be the unique equilibrium solution under the prescribed boundary conditions.

5. Kinetic stability analysis

In the normal mode approach, we look for a solution of the form

$$\mathbf{u}(\mathbf{X}) = \mathbf{z}(y_2)\mathrm{e}^{\mathrm{i}y_1} + C.C., \quad y_1 = s = \mathbf{K}\cdot\mathbf{X}, \quad y_2 = \mathbf{N}\cdot\mathbf{X}, \tag{5.1}$$

where $\mathbf{z}$ is to be determined and we expect that $\mathbf{z}$ will have two separate expressions valid for $0 < y_2 < \infty$ and $-\infty < y_2 < 0$, respectively. The $C.C.$ in (5.1) denotes the complex conjugate of the preceding term. Equation (4.13) yields

$$T\mathbf{z}'' + \mathrm{i}(R + R^T)\mathbf{z}' - Q\mathbf{z} = \mathbf{0}, \tag{5.2}$$

where a prime now denotes differentiation with respect to the argument y_2 and the components of the 2×2 matrices T, R, Q are given by

$$T_{ij} = E_{iAjB} N_A N_B, \quad R_{ij} = E_{iAjB} K_A N_B, \quad Q_{ij} = E_{iAjB} K_A K_B. \tag{5.3}$$

These expressions are valid for both phases. To specialize to a specific phase, we only need to put a superscript/subscript "+" or "−" on the relevant quantities such as T_{ij} and E_{iAjB}.

To find the general solution of (5.2), let

$$\mathbf{z}(y_2) = \mathbf{a} e^{ipy_2}, \tag{5.4}$$

where p is a constant and $\mathbf{a}$ a constant vector. On substituting (5.4) into (5.2), we obtain

$$\left\{ p^2 T + (R + R^T) p + Q \right\} \mathbf{a} = \mathbf{0}. \tag{5.5}$$

Since we have assumed that the materials in the "+" and "−" phases both satisfy the strong ellipticity condition, it follows that the eigenvalues determined by (5.5) cannot be pure real. Denote the four eigenvalues by

$$p_1, \ p_2, \ \hat{p}_1, \ \hat{p}_2, \quad \text{where } \mathrm{Im}(p_1) > 0, \ \mathrm{Im}(p_2) > 0,$$

and the associated eigenvectors by

$$\mathbf{a}^{(1)}, \ \mathbf{a}^{(2)}, \ \hat{\mathbf{a}}^{(1)}, \ \hat{\mathbf{a}}^{(2)},$$

where here and hereafter a hat signifies complex conjugation and $\mathrm{Im}(p_1)$ denotes the imaginary part of p_1. The general solution for $\mathbf{z}$ is then given by

$$\mathbf{z} = \begin{cases} \displaystyle\sum_{k=1}^{2} d_k^- \mathbf{a}_-^{(k)} e^{ip_k^- y_2}, & 0 < y_2 < \infty, \\ \displaystyle\sum_{k=1}^{2} d_k^+ \hat{\mathbf{a}}_+^{(k)} e^{i\hat{p}_k^+ y_2}, & -\infty < y_2 < 0, \end{cases} \tag{5.6}$$

where d_k^+, d_k^- $(k = 1, 2)$ are constants and p_k^+ and $\hat{p}_k^-$ $(k = 1, 2)$ have been selected to ensure that $\mathbf{z} \to 0$ as $y_2 \to \pm\infty$. Hence,

$$\mathbf{z}^- = \sum_{k=1}^{2} d_k^- \mathbf{a}_-^{(k)} = A^- \mathbf{d}^-, \tag{5.7}$$

$$\mathbf{z}^+ = \sum_{k=1}^{2} d_k^+ \hat{\mathbf{a}}_+^{(k)} = \hat{A}^+ \mathbf{d}^+, \tag{5.8}$$

where

$$A^{\pm} = (\mathbf{a}_{\pm}^{(1)}, \mathbf{a}_{\pm}^{(2)}), \quad \mathbf{d}^{\pm} = (d_1^{\pm}, d_2^{\pm})^T. \tag{5.9}$$

With the use of (5.1), we obtain

$$(E_{iAjB}^+ u_{j,B}^+ N_A) = \left(T^+ \mathbf{z}'^+ + iR_+^T \mathbf{z}^+ \right) e^{iy_1} + C.C. = \hat{M}^+ \mathbf{z}^+ e^{iy_1} + C.C., \tag{5.10}$$

$$(E_{iAjB}^- u_{j,B}^- N_A) = \left(T^- \mathbf{z}'^- + iR_-^T \mathbf{z}^- \right) e^{iy_1} + C.C. = -M^- \mathbf{z}^- e^{iy_1} + C.C., \tag{5.11}$$

where the notation $(\triangle_i)$ is used to mean the vector with components $\triangle_i$ and the second relation in each equation serves as the definition for the *surface impedance tensors* M^+ and M^-, respectively; see Fu and Mielke (2002). With the use of (5.6), we obtain

$$M^{\pm} = -iB^{\pm} A_{\pm}^{-1}, \quad B^{\pm} = (\mathbf{b}_{\pm}^{(1)}, \ \mathbf{b}_{\pm}^{(2)}), \tag{5.12}$$

where

$$\mathbf{b}_{\pm}^{(k)} = (p_k^{\pm} T^{\pm} + R_{\pm}^T) \mathbf{a}_{\pm}^{(k)}, \quad k = 1, 2. \tag{5.13}$$

It is known (see, e.g., Mielke and Fu (2004)) that the surface impedance tensor M (=M^+ or M^-) is Hermitian and satisfies the matrix Riccati equation

$$(M - \mathrm{i}R)T^{-1}(M + \mathrm{i}R^T) - Q = 0. \tag{5.14}$$

The simple matrix equation (5.14) can be solved either exactly (when it is possible) or numerically to find M. We assume that the strain-energy function is locally convex at $\lambda_1 = \lambda_2 = 1$. It can then be shown that M must necessarily be positive definite at $\lambda_1 = \lambda_2 = 1$ and that equation (5.14) has a unique solution having this property. The unique solution can then be extended to other values of λ_1 and λ_2. This provides a selection criterion when solving (5.14). For instance, when $N_1 = 0, N_2 = 1$, the principal axes of stretch in both phases are aligned with the coordinate axes. The three matrices T, R and Q then take the simple form

$$\mathbf{T} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \tag{5.15}$$

where

$$T_1 = E_{1212}, \;\; T_2 = E_{2222}, \;\; R_1 = E_{1122}, \;\; R_2 = E_{2112}, \;\; Q_1 = E_{1111}, \;\; Q_2 = E_{2121}.$$

In this case, the unique solution of (5.14) having the above-mentioned property is given by

$$M = \begin{pmatrix} M_1 & M_3 + \mathrm{i}M_4 \\ M_3 - \mathrm{i}M_4 & M_2 \end{pmatrix}, \tag{5.16}$$

where

$$M_1 = \sqrt{T_1 Q_1 - \frac{T_1}{T_2}\left(\frac{R_1 + R_2}{1 + r}\right)^2}, \quad r = \sqrt{T_1 Q_2/(T_2 Q_1)},$$

$$M_2 = r\frac{T_2}{T_1}M_1, \quad M_3 = 0, \quad M_4 = \frac{rR_1 - R_2}{1 + r}. \tag{5.17}$$

Return now to the general case. On substituting (4.4) and (5.1) into (4.9), we obtain

$$\gamma\bar{\mathbf{f}} + [\mathbf{z}] = 0, \tag{5.18}$$

whereas with the use of (4.4), (5.1), (5.10) and (5.11), the jump condition (4.18) becomes

$$\hat{M}^+\mathbf{z}^+ + M^-\mathbf{z}^- - \mathrm{i}\gamma\boldsymbol{\beta} = 0, \tag{5.19}$$

where

$$\boldsymbol{\beta} = [\bar{\boldsymbol{\pi}}]\mathbf{K}. \tag{5.20}$$

Equation (5.18) and (5.19) are two linear equations for $\mathbf{z}^+$ and $\mathbf{z}^-$. Eliminating $\mathbf{z}^-$, we obtain

$$P\mathbf{z}^+ = -\gamma(M^-\bar{\mathbf{f}} - \mathrm{i}\boldsymbol{\beta}), \quad \mathbf{z}^- = \mathbf{z}^+ + \gamma\bar{\mathbf{f}}, \tag{5.21}$$

where P, defined by

$$P = \hat{M}^+ + M^-, \tag{5.22}$$

may be referred to as the *interfacial impedance tensor*. It is then seen that if $\det P = 0$, a standing-wave solution with $\mathbf{z}^- = \mathbf{z}^+ \neq \mathbf{0}$, $\gamma = 0$ exists, which implies that the pairwise homogeneous state may bifurcate into an inhomogeneous two-phase state with $\gamma = 0$. It will be shown in the next section that the bifurcation condition $\det P = 0$ also defines a stability boundary with respect to a certain class of perturbations.

We remark that in the jargon of theory of partial differential equations, $\det P = 0$ signifies violation of the so-called *complementing condition* (Simpson and Spector 1989, 1991, Renardy and Rogers 1992). Satisfaction of this condition at an interface is as important as that of the ellipticity condition away from the interface for the well-posedness of boundary value problems. If two joined-half-spaces are in a state with this condition violated, the mathematical problem governing their further incremental deformations is ill-posed. Although physically we may argue that $\det P = 0$ simply signifies onset of interfacial wrinkling, it is not yet clear how post-wrinkling states could be determined; see Ogden and Fu (1996) and Fu (1999) for a discussion of the corresponding half-space problem.

If, on the other hand, det $P \neq 0$, the unique solution of $(5.21)_1$ is given by

$$\mathbf{z}^+ = -\gamma P^{-1}(M^-\bar{\mathbf{f}} - \mathrm{i}\boldsymbol{\beta}). \tag{5.23}$$

Once $\mathbf{z}^+$ and $\mathbf{z}^-$ have been determined, we may evaluate (4.24). With the use of (4.4) and (5.1), it takes the form (4.6) with L given by

$$\gamma L = \mathrm{i}[\mathbf{z}\cdot\bar{\boldsymbol{\pi}}\mathbf{K}] + \bar{\mathbf{f}}\cdot\left(M^-\mathbf{z}^- + \mathrm{i}\gamma\bar{\boldsymbol{\pi}}^-\mathbf{K}\right). \tag{5.24}$$

With the use of (5.20), (5.21) and (5.23), this expression can be reduced to

$$L = \bar{\mathbf{f}}\cdot M^-\bar{\mathbf{f}} - \hat{\mathbf{g}}\cdot P^{-1}\mathbf{g}, \quad \text{where } \mathbf{g} = M^-\bar{\mathbf{f}} - \mathrm{i}\boldsymbol{\beta}. \tag{5.25}$$

Since M^+, M^-, and hence P are all Hermitian, the above expression shows that L is always real. According to the kinetic stability criterion, the pairwise homogeneous deformation is stable if $L > 0$ and unstable if $L < 0$. The boundary of stability $L = 0$ gives us another bifurcation condition.

6. Stability analysis by the energy criterion

We first consider a general finite elastic body that occupies the region Ω in its reference configuration. We assume that $\partial\Omega = S_u \bigcup S_t$ where S_u is part of $\partial\Omega$ where displacement is prescribed and S_t is where a dead-load surface traction $\bar{\mathbf{t}}$ is prescribed. The resulting displacement field is a two-phase deformation with Ω divided into a core region Ω_+ and an outer region $\Omega_- = \Omega\backslash\Omega_+$, the interface being denoted by S_p. We assume that the (primary) deformation is given by (2.1) and the interface is defined by

$$\mathbf{X} = \mathbf{Y}(\boldsymbol{\alpha}), \tag{6.1}$$

where $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}^T$ parametrizes the interface. The total energy corresponding to this two-phase deformation is then given by

$$E_0 = \int_{\Omega_+\bigcup\Omega_-} W(\bar{\mathbf{F}})dX_1dX_2dX_3 - \int_{S_t}\bar{\mathbf{t}}\cdot\mathbf{x}dS. \tag{6.2}$$

Now consider a perturbation of the interface (6.1) and a perturbation of the displacement field (2.1) defined by

$$\mathbf{X} = \mathbf{Y}(\boldsymbol{\alpha}) + \epsilon\boldsymbol{\xi}^{(1)}(\mathbf{Y}) + \epsilon^2\boldsymbol{\xi}^{(2)}(\mathbf{Y}) + O(\epsilon^3), \tag{6.3}$$

and

$$\tilde{\mathbf{x}} = \mathbf{x}(\mathbf{X}) + \epsilon\boldsymbol{u}^{(1)}(\mathbf{X},t) + \epsilon^2\boldsymbol{u}^{(2)}(\mathbf{X},t) + O(\epsilon^3), \tag{6.4}$$

respectively, where ϵ is a small parameter characterizing the size of the perturbations. The inclusion of $O(\epsilon^2)$ terms in (6.3) and (6.4) is usually not necessary for elasticity problems without phase transformations. But for the present problem under consideration they are necessary in order to ensure that the $\tilde{\mathbf{x}}$ given by (4.8) is continuous even at the next order. More precisely, the interface in the current configuration is given by

$$\begin{aligned}\tilde{x}_i &= x_i(\mathbf{Y} + \epsilon\boldsymbol{\xi}^{(1)} + \epsilon^2\boldsymbol{\xi}^{(2)} + \cdots) + \epsilon u_i^{(1)}(\mathbf{Y} + \epsilon\boldsymbol{\xi}^{(1)} + \cdots) + \epsilon^2 u_i^{(2)}(\mathbf{Y} + \cdots) + \cdots \\ &= x_i + \epsilon\left\{\bar{F}_{iA}\xi_A^{(1)} + u_i^{(1)}\right\} + \epsilon^2\left\{u_i^{(2)} + u_{i,A}^{(1)}\xi_A^{(1)} + \bar{F}_{iA}\xi_A^{(2)}\right. \\ &\quad \left. + \frac{1}{2}\bar{F}_{iA,B}\xi_A^{(1)}\xi_B^{(1)}\right\} + \cdots,\end{aligned} \tag{6.5}$$

where all the quantities in the last expression are evaluated at $\mathbf{X} = \mathbf{Y}(\alpha)$. It then follows from the continuity condition $[\tilde{x}_i] = 0$ that

$$[u_i^{(1)}] + [\bar{F}_{iA}]\xi_A^{(1)} = 0, \tag{6.6}$$

$$[u_i^{(2)}] + [\bar{F}_{iA}]\xi_A^{(2)} + [u_{i,A}^{(1)}]\xi_A^{(1)} + \frac{1}{2}[\bar{F}_{iA,B}]\xi_A^{(1)}\xi_B^{(1)} = 0. \tag{6.7}$$

It is then seen that if $u_i^{(2)} = 0, \quad \xi_A^{(2)} = 0$, the continuity condition (6.7) in general cannot be satisfied.

The energy corresponding to the perturbed state is given by

$$E_1 = \int_{\Omega'_+ \bigcup \Omega'_-} W(\mathbf{F}) dX'_1 dX'_2 dX'_3 - \int_{S_t} \bar{\mathbf{t}} \cdot \tilde{\mathbf{x}} dS, \tag{6.8}$$

where Ω'_+ is the region enclosed by the new interface (6.3), $\Omega'_- = \Omega \backslash \Omega'_+$, and from (6.4) the deformation gradient $\mathbf{F}$ is given by

$$F_{iA} = \tilde{x}_{i,A} = \bar{F}_{iA} + \epsilon u_{i,A}^{(1)} + \epsilon^2 u_{i,A}^{(2)} + \cdots . \tag{6.9}$$

In order to evaluate the energy increment $E_1 - E_0$, we use a coordinate transformation

$$\mathbf{X}' = \mathbf{X} + \epsilon \boldsymbol{\phi}^{(1)}(\mathbf{X}) + \epsilon^2 \boldsymbol{\phi}^{(2)}(\mathbf{X}) + \cdots \tag{6.10}$$

to transform the regions of integration Ω'_+ and Ω'_- to Ω_+ and Ω_-, respectively. It is seen that for this purpose the unknown functions $\boldsymbol{\phi}^{(i)}$ $(i = 1, 2)$ need to satisfy the condition

$$\boldsymbol{\phi}^{(i)}(\mathbf{X}) = \begin{cases} 0 & \text{on } \partial\Omega, \\ \boldsymbol{\xi}^{(i)}(\mathbf{Y}) & \text{on } \mathbf{X} = \mathbf{Y}(\boldsymbol{\alpha}). \end{cases} \tag{6.11}$$

We also require $\boldsymbol{\phi}^{(i)}(\mathbf{X})$ to be smooth in the sense that $\phi_A^{(i)}$, $\phi_{A,B}^{(i)}$ are both continuous across $\mathbf{X} = \mathbf{Y}(\alpha)$, and

$$\phi_A^{(i)} = \xi_A^{(i)}, \quad \phi_{A,B}^{(i)} = \xi_{A,B}^{(i)} \quad \text{on } \mathbf{X} = \boldsymbol{Y}(\boldsymbol{\alpha}). \tag{6.12}$$

The energy increment due to the perturbations (6.3) and (6.4) is then given by

$$\triangle E = E_1 - E_0 = \int_{\Omega_+ \bigcup \Omega_-} \left\{ W(\mathbf{F}) \left| \frac{\partial \mathbf{X}'}{\partial \mathbf{X}} \right| - \mathbf{W}(\bar{\mathbf{F}}) \right\} dX_1 dX_2 dX_3 - \int_{S_t} \bar{\mathbf{t}} \cdot (\tilde{\mathbf{x}} - \mathbf{x}) dS, \tag{6.13}$$

where $|\partial \mathbf{X}' / \partial \mathbf{X}|$ is the Jacobian corresponding to the coordinate transformation (6.10) and is given by

$$\left| \frac{\partial \mathbf{X}'}{\partial \mathbf{X}} \right| = 1 + \epsilon \phi_{A,A}^{(1)} + \epsilon^2 \left\{ \phi_{A,A}^{(2)} + \frac{1}{2} (\phi_{A,A}^{(1)})^2 - \frac{1}{2} \phi_{A,B}^{(1)} \phi_{B,A}^{(1)} \right\} + O(\epsilon^3). \tag{6.14}$$

We observe that the $W(\mathbf{F})$ in (6.13) is evaluated at $\boldsymbol{X}'$ given by (6.10) and should be expanded in the neighborhood of $\mathbf{X}' = \mathbf{X}$ in the evaluation of (6.13). The rest of the derivation is straightforward. It involves the usual application of the divergence theorem to convert volume integrals to surface integrals, followed by the use of the continuity condition (6.7) to eliminate the second order fields $[\mathbf{u}^{(2)}]$ and $\boldsymbol{\xi}^{(2)}$. It can be shown that to order ϵ^2 the energy increment is given by

$$\triangle E = \frac{1}{2} \int_{S_p} P dS, \tag{6.15}$$

where

$$\begin{aligned} P = [E_{iAjB} u_{j,B} u_i] N_A + 2[\bar{\pi}_{i,A} u_{i,A}] \Gamma - 2\bar{\pi}_{i,A} N_A [u_{i,B}] \xi_B \\ + [\bar{W}](\Gamma \xi_{B,B} - \xi_{A,B} N_A \xi_B) + [\bar{W}_{,B}] \xi_B \Gamma - [\bar{\pi}_{i,A} \bar{F}_{iB,C}] N_A \xi_B \xi_C, \end{aligned} \tag{6.16}$$

and we have replaced $\epsilon \mathbf{u}^{(1)}$ and $\epsilon \boldsymbol{\xi}^{(1)}$ by $\mathbf{u}$ and $\boldsymbol{\xi}$, respectively, so that $\mathbf{u}, \boldsymbol{\xi}$ and Γ here have the same meaning as in the previous section. In obtaining (6.15) we have also assumed that the perturbations satisfy the equilibrium equation (4.13).

The right hand side of (6.15) should be half the second variation of the energy functional (6.8). Thus, it can also be derived by the usual method of taking the second variation of (6.8). We have verified that this is indeed the case. We have also checked that the expression is consistent with Grinfeld's (1991) equation (4.1.10).

For the plane-strain problem considered in the previous two sections, equation (6.15) yields

$$\triangle E = \frac{1}{4\pi} \int_0^{2\pi} \left\{ [\bar{E}_{iAjB} u_i u_{j,B}] N_A + 2[\bar{\pi}_{iA} u_{i,A}] \Gamma - 2\bar{\pi}_{iA} N_A [u_{i,B}] \xi_B \right.$$

$$+[\bar{W}](\Gamma\xi_{A,A} - \xi_{A,B}N_A\xi_B)\big\}\,dy_1, \tag{6.17}$$

where the energy increment is evaluated and averaged over one period of the perturbation in the y_1-direction.

With the use of (4.4), (5.1), (5.10) and (5.11), we first obtain

$$\triangle E = \hat{\mathbf{z}}^+ \cdot \hat{M}^+\mathbf{z}^+ + \hat{\mathbf{z}}^- \cdot M^-\mathbf{z}^- + 2\mathrm{Re}\,\{i[\mathbf{z}\cdot\bar{\boldsymbol{\pi}}\mathbf{K}]\hat{\gamma}\}\,. \tag{6.18}$$

Using the displacement continuity condition $(5.21)_2$ to eliminate $\mathbf{z}^-$ in favor of $\mathbf{z}^+$, and then (5.20), (5.22) and $(5.25)_2$ to simplify the resulting expression further, we finally arrive at

$$\triangle E = \hat{\mathbf{w}} \cdot H\mathbf{w}, \tag{6.19}$$

where

$$H = \begin{pmatrix} P & \mathbf{g} \\ \hat{\mathbf{g}}^T & \bar{\mathbf{f}}\cdot M^-\bar{\mathbf{f}} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \mathbf{z}^+ \\ \gamma \end{pmatrix}.$$

Note that in obtaining (6.19) we have not imposed the traction continuity condition $(5.21)_1$. Thus, the perturbations are only required to satisfy the equilibrium equation, the displacement continuity condition and the decaying condition at infinity.

If $\triangle E$ is positive definite, the pairwise homogeneous deformation is a local energy minimizer and the two-phase state is then said to be stable. It is seen that a necessary condition for $\triangle E$ to be positive definite is that P is positive definite. Under this necessary condition, the matrix H can be written as

$$H = \begin{pmatrix} P^{\frac{1}{2}} & 0 \\ (\hat{P}^{-\frac{1}{2}}\hat{\mathbf{g}})^T & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & L \end{pmatrix}\begin{pmatrix} P^{\frac{1}{2}} & P^{-\frac{1}{2}}\mathbf{g} \\ 0 & I \end{pmatrix}, \tag{6.20}$$

where L is given by $(5.25)_1$. It then becomes clear that the necessary and sufficient conditions for $\triangle E$ to be positive definite are

$$\text{(i) } P \text{ is positive definite, and (ii) } L > 0. \tag{6.21}$$

If we assume $\Gamma \equiv 0$, then the present problem reduces to a stability problem for two half-spaces of different materials that are bonded together (a joint-body problem, as discussed by e.g. Simpson and Spector 1991). Equation (6.18) reduces to

$$\triangle E = \hat{\mathbf{z}}^+ \cdot P\mathbf{z}^+. \tag{6.22}$$

Thus, the first condition (i) above is a sufficient condition for stability when the pairwise homogeneous deformation is viewed as a joint-body problem. This result is consistent with that of Simpson and Spector (1991) who also showed that a necessary condition for stability is that P is positive semi-definite.

When $\Gamma = 0$, the traction continuity condition $(5.21)_1$ reduces to

$$P\mathbf{z}^+ = 0, \tag{6.23}$$

which has a non-trivial solution only if

$$\det P = 0. \tag{6.24}$$

This is the bifurcation condition for a straight interface to wrinkle. Such a condition, in less general form, has been derived by a number of authors; see, e.g. Dowaikh and Ogden (1991). Equation (6.24) can be obtained from the secular equation for interfacial (Stoneley) waves by setting the wave speed to zero. Existence and uniqueness of interfacial waves has been the subject of many studies; see, e.g., Barnett *et al.* (1985), Chadwick and Borejko (1994), Chadwick (1995).

Since the present problem does not have a natural lengthscale, a general perturbation, that is periodic along the interface with unit period, can be represented as

$$\mathbf{u} = \sum_{n=-\infty}^{\infty} a_n\mathbf{z}(ny_2)\mathrm{e}^{iny_1}, \quad \Gamma = \sum_{n=-\infty}^{\infty} b_n\mathrm{e}^{iny_1}, \tag{6.25}$$

where

$$a_0 = 0, \;\; b_0 = 0, \;\; a_{-n} = \hat{a}_n, \;\; \mathbf{z}(-ny_2) = \hat{\mathbf{z}}(ny_2), \;\; b_{-n} = \hat{b}_n,$$

and we reiterate that a superimposed hat signifies complex conjugation. By its construction, the perturbation given by (6.25) satisfies the equilibrium equation, the displacement continuity conditions at the interface and decays to zero as $y_2 \to \pm\infty$. The expression (6.17) is still valid for such a general perturbation. It can then be shown that the energy increment, denoted by $(\triangle E)_g$, due to this general perturbation is given by

$$(\triangle E)_g = \sum_{n=1}^{\infty} n\hat{\mathbf{w}}^{(n)} \cdot H\mathbf{w}^{(n)}, \quad \text{where } \mathbf{w}^{(n)} = \begin{pmatrix} a_n \mathbf{z}^+ \\ b_n \end{pmatrix}. \tag{6.26}$$

It is then seen that the necessary and sufficient condition for $(\triangle E)_g$ to be positive definite is that the Hermitian matrix H is positive definite, same as for $\triangle E$. Thus, the pair-wise homogeneous deformation is stable with respect to general periodic perturbations if conditions (6.21) are satisfied, and unstable if either P has at least one negative eigenvalue or L is negative. At the stability boundary $\det P = 0$ or $L = 0$, higher-order terms in the expansion of the energy increment are needed to establish stability/instability.

7. Numerical results

We shall consider the case when the strain-energy function is given by (3.7). The PTZ analysis presented in Section 3 shows that following any loading path starting from the stress-free state, the new phase boundary has either $N_1^2 = 0$ or $N_1^2 = 1$. Without loss of generality, we assume that $N_1 = 0, N_2 = 1$. In this case, the principal axes of stretch in both phases are aligned with the coordinate axes and we have

$$\bar{f}_i = [\lambda_2]\delta_{2i}, \quad [z_i] = -\gamma[\lambda_2]\delta_{2i}, \quad \beta_i = [\bar{\pi}_{11}]\delta_{i1}. \tag{7.1}$$

The elastic moduli defined by (4.12) are given by

$$E_{iAjB} = 2dF_{jB}F_{iA} + (c + dI_1)\delta_{ij}\delta_{AB} - d(\delta_{ij}F_{mB} + \delta_{mj}F_{iB})F_{mA}$$
$$-dB_{im}\delta_{jm}\delta_{AB} + (J\phi')'\, JF^{-1}_{Bj}F^{-1}_{Ai} - J\phi' F^{-1}_{Aj}F^{-1}_{Bi}, \tag{7.2}$$

where the right hand side is evaluated at the primary deformation. From equations (6.3.30) and (6.3.31) of Ogden (1984), we deduce that the strong ellipticity condition is satisfied if

$$c + d > 0, \quad \frac{\partial^2 W}{\partial \lambda_1^2} = c + d + (d + \phi'')\lambda_2^2 > 0, \quad \frac{\partial^2 W}{\partial \lambda_2^2} = c + d + (d + \phi'')\lambda_1^2 > 0. \tag{7.3}$$

We have seen already that violation of the third inequality above is necessary and sufficient for the existence of a phase boundary with $N_A = \delta_{A2}$ across which λ_2 suffers a jump. The corresponding PTZ is given by (3.50). Likewise, violation of the second inequality in (7.3) is necessary and sufficient for the existence of a phase boundary with $N_A = \delta_{A1}$ across which λ_1 suffers a jump. The corresponding PTZ is given by (3.51). We note that the non-ellipticity boundaries are given by $\partial^2 W/\partial\lambda_1^2 = 0$ and $\partial^2 W/\partial\lambda_2^2 = 0$, that is by

$$\lambda_2 = \frac{1}{\lambda_1}\left\{J_c \pm \frac{1}{\sqrt{3}}\sqrt{a_1 - d - \frac{c+d}{\lambda_1^2}}\right\}, \quad \text{and } \lambda_1 = \frac{1}{\lambda_2}\left\{J_c \pm \frac{1}{\sqrt{3}}\sqrt{a_1 - d - \frac{c+d}{\lambda_2^2}}\right\}. \tag{7.4}$$

By comparing $(7.4)_{1,2}$ with (3.50) and (3.51), respectively, we see that the two non-ellipticity boundaries are embedded in the corresponding PTZs.

As an illustrative example, we choose $a_1 = 0.8, d = 0.2, J_c = 2, c = 0.032$. The nose of the PTZ curve in Fig.2 is at $\lambda_1 = \sqrt{(c+d)/(a_1-d)} = 0.62183$. The lower (respectively upper) branch of the non-ellipticity boundary $(7.4)_2$ intersects the lower (respectively upper) branch of the PTZ curve (3.50) at $\lambda_1 = 0.74555$ (respectively $\lambda_1 = 0.75463$), and over the interval $0.62183 \le \lambda_1 \le 0.75463$ although a two-phase state is mathematically possible, the deformation in one of the two phases violates the strong ellipticity condition. Thus, our calculations will focus on the interval $0.75463 < \lambda_1 < \infty$.

Our numerical computations show that $\det P = 0$ when $\lambda_1 = 0.756245$ and that P is positive definite for $\lambda_1 > 0.756245$. At this critical value of λ_1, we have $\triangle E \to \infty$, as can be seen from

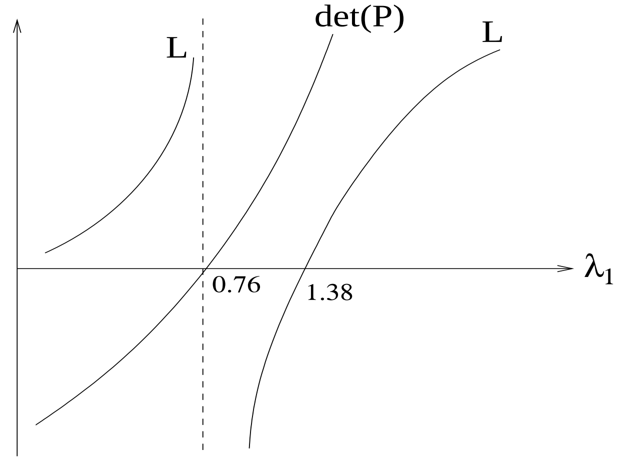

Figure 5. An artistic impression of the signs of $\det P$ and $\triangle E$.

(5.25). Fig.5 shows an artistic impression of the signs of $\det P$ and L. The conditions (6.21) are satisfied, and hence the pairwise homogeneous deformation is stable, only for $\lambda_1 > 1.38154$, which is only a subset of the λ_1 values (namely $\lambda_1 > 0.756245$) over which the pairwise homogeneous deformation is stable when viewed as a joint-body problem.

We observe from Fig.2 that for the present example, if we follow a loading path of biaxial tension ($\lambda_1 = \lambda_2$), then a phase transformation will take place when $\lambda_1 = 1.1602$. Fig. 5 shows that the resulting pairwise homogeneous deformation would be unstable. However, the above analysis indicates that the two-phase deformation will be stable as long as the loading path hits the lower branch of the PTZ curve in Fig.2 beyond the critical value $\lambda_1 = 1.38154$.

8. Conclusion

In this paper we have considered the bifurcation and stability of two joined half-spaces that correspond to two different phases of a single material. A necessary condition for such a two-phase deformation to exist at all is that the strain-energy function violates the strong ellipticity condition at some deformation gradient. Our characterization of the two-phase deformation follows the method of PTZ construction. One of our objectives has been to explain this general method through a simple example.

Once the two-phase homogeneous deformation is determined, our next objective has been to investigate whether this deformation is unique under the same boundary conditions. Our analysis follows the traditional method of adjacent equilibria. The incremental deformation is sinusoidal along the interface and decays exponentially away from the interface. It is found that two types of bifurcations are possible, with bifurcation conditions given by $\det P = 0$ and $\det P \neq 0$ but $L = 0$, respectively. When $\det P = 0$ is satisfied, the bifurcation solution has $\Gamma = 0$, whereas when $\det P \neq 0$ but $L = 0$, the bifurcation solution has $\Gamma \neq 0$.

Stability is the other issue which we address concerning the two-phase homogeneous deformation. We use two stability criteria. The first is a kinetic stability criterion based on a quasi-static approach: the two-phase state is unstable if a perturbation of the interface in the undeformed configuration grows and stable otherwise. The other criterion is the energy criterion. We simply compare the total energies corresponding to the two-phase homogeneous deformation and to the perturbed inhomogeneous deformation. The perturbations considered are the same normal modes as those in the bifurcation analysis, and consist of perturbations of the interface in both the current and undeformed configurations. We give an explicit formula for the energy difference which can be used to determine whether the two-phase homogenous deformation is an energy minimizer with respect to the class of perturbations considered. In the application of both criteria, we require the perturbations to satisfy the equilibrium condition, the displacement continuity condition and the decaying condition at infinity. In the application of the kinetic stability criterion we additionally require the perturbations to satisfy the traction continuity condition (in fact equation (4.3) makes sense only if $\boldsymbol{\pi}\tilde{\mathbf{N}}$ is continuous). The kinetic stability criterion breaks down when $\det P = 0$, and we see that it coincides with the energy criterion only if it is supplemented by the condition that P

is positive definite. For the numerical example shown in Fig.5, the kinetic stability criterion would wrongly predict that the two-phase deformation is stable for $\lambda_1 < 0.76$.

Throughout our paper, we have used the Hadamard strain-energy function in our illustrative numerical calculations. An important property of this class of strain energy function is that when the material is loaded from the stress-free state, then when a phase transformation takes place the new phase boundary must necessarily be aligned with one of the principal axes of stretch. In this case, the surface impedance tensors have explicit expressions and our numerical calculations of the bifurcation condition and the energy difference are relatively easy. We remark, however, that there exist strain-energy functions for which the phase boundary will not be aligned with any of the principal axes of stretch. In this case although explicit expressions for the surface impedance tensors are not possible, our formulae are very amenable to numerical calculations. Thus, the present paper provides a general framework under which multi-phase deformations corresponding to any specific energy function can be characterized and their bifurcation/stability properties assessed.

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